MATH 220
3 March 2025

WARM-UP: What two properties must a linear transformation satisfy?

LINEAR TRANSFORMATION:

A function $T: \underbrace{\mathbb{R}^m}_{\text{domain}} \to \underbrace{\mathbb{R}^n}_{\text{codomain}}$ that satisfies

(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u},\vec{v} \in \mathbb{R}^m$ ← addition

in

(b) $T(r\cdot\vec{u}) = r\cdot T(\vec{u})$ for all $\vec{u}\in\mathbb{R}^m$ and $\underbrace{r\in\mathbb{R}}_{r \text{ is a scalar}}$ ← scalar multiplication

Example: $T:\mathbb{R}^2\to\mathbb{R}$ given by $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = x_1+x_2$

check linear:

ADDITION:

$$T\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}+\begin{bmatrix}v_1\\v_2\end{bmatrix}\right) = T\left(\begin{bmatrix}u_1+v_1\\u_2+v_2\end{bmatrix}\right) = u_1+v_1+u_2+v_2$$

same

$$T\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}\right)+T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right) = (u_1+u_2)+(v_1+v_2) = u_1+u_2+v_1+v_2$$

SCALAR MULTIPLICATION:

$$T\left(r\cdot\begin{bmatrix}u_1\\u_2\end{bmatrix}\right) = T\left(\begin{bmatrix}ru_1\\ru_2\end{bmatrix}\right) = ru_1+ru_2$$

same

$$r\cdot T\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}\right) = r\cdot(u_1+u_2) = ru_1+ru_2$$

Linear Algebra – Day 11

MATH 220

1. *Re-enact the following dialogue with your group.*

Milo: Hey, Maura! I have this function $T : \mathbb{R}^3 \to \mathbb{R}^2$ that has the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2 \\ x_2 + x_3 \end{bmatrix}.$$

Maura: That function is *not* a linear transformation!

$T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 6 \\ 7 \end{bmatrix} = T(\vec{u} + \vec{v})$

Milo: But why? I looks like a wonderful function.

Maura: We can find an example of vectors $\mathbf{u}$ and $\mathbf{v}$ where $T(\mathbf{u}+\mathbf{v})$ does not equal $T(\mathbf{u})+T(\mathbf{v})$.

Group discussion: Try to find two specific vectors $\mathbf{u}$ and $\mathbf{v}$ such that $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u})+T(\mathbf{v})$.

$T\left(\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{u}\right) = \begin{bmatrix} 2(1)+2 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ $\quad$ $T\left(\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{v}\right) = \begin{bmatrix} 2(1)+2 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\quad$ $T\left(\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{u+v}\right) = T\left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2(2)+2 \\ 3+4 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$

Milo: This function actually fails *both* requirements of a linear transformation! I can find an example of a vector $\mathbf{u}$ and a scalar c where $T(c \cdot \mathbf{u})$ does not equal $c \cdot T(\mathbf{u})$.

Group discussion: Try to find a specific vector $\mathbf{u}$ and scalar c such that $T(c \cdot \mathbf{u}) \neq c \cdot T(\mathbf{u})$.

$$T\left(5 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2(10)+2 \\ 5+0 \end{bmatrix} = \begin{bmatrix} 22 \\ 5 \end{bmatrix} \neq 5 \cdot T\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = 5\begin{bmatrix} 2(2)+2 \\ 1+0 \end{bmatrix} = 5\begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 \\ 5 \end{bmatrix}$$

2. Now consider at the function $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by the formula $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_2 + x_3 \end{bmatrix}$.

(a) Use the formula above to calculate $T\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}\right)$. $= \begin{bmatrix} 2(a_1+b_1) \\ (a_2+b_2)+(a_3+b_3) \end{bmatrix} = \begin{bmatrix} 2a_1 + 2b_1 \\ a_2+b_2+a_3+b_3 \end{bmatrix}$

same!

(b) Use the formula above to individually calculate $T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) + T\left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right)$. $= \begin{bmatrix} 2a_1 \\ a_2+a_3 \end{bmatrix} + \begin{bmatrix} 2b_1 \\ b_2+b_3 \end{bmatrix} = \begin{bmatrix} 2a_1+2b_1 \\ a_2+a_3+b_2+b_3 \end{bmatrix}$

(c) Are the results of parts (a) and (b) equal? Yes!

(d) Use the formula above to calculate $T\left(\begin{bmatrix} c \cdot a_1 \\ c \cdot a_2 \\ c \cdot a_3 \end{bmatrix}\right)$. $= \begin{bmatrix} 2ca_1 \\ ca_2 + ca_3 \end{bmatrix}$

Same!

(e) Use the formula above to calculate $c \cdot T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right)$. $= c\begin{bmatrix} 2a_1 \\ a_2+a_3 \end{bmatrix} = \begin{bmatrix} 2ca_1 \\ ca_2+ca_3 \end{bmatrix}$

(f) Are the results of parts (d) and (e) equal? Yes!

3. Here is the formula for a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 3x+7y \\ -y \end{bmatrix}.$$

☝ I already checked to make sure T follows the two requirements!

(a) Using the formula for T above, calculate $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, and $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2+2(3) \\ 3(2)+7(3) \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ 27 \\ -3 \end{bmatrix}$

(b) **Ava:** I figured out that $\quad T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$.

☝ REMEMBER: linear transformations don't mess with linear combinations.

Jason: Well done, Ava! That is correct.

Ava: But I don't need the formula to figure out $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.

Jason: What do you mean? Are you a magician?

Ava: No, silly! I just used the work I already did and got

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so:

$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 2 \cdot \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}. = \begin{bmatrix} 8 \\ 27 \\ -3 \end{bmatrix}$$

Group chat: what did Ava do to figure out $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$?

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \\ a_{31}x + a_{32}y \end{bmatrix}$$

(c) Try to come up with a 3×2 matrix A for which $T(\mathbf{x}) = A\mathbf{x}$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \\ 0 & -1 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+7y \\ 0x-y \end{bmatrix}$

4. Let $A = \begin{bmatrix} 2 & 0 & -2 \\ -1 & 3 & 4 \end{bmatrix}$ and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Is the vector $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$ in the range of T?

Is there some $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$?

Yes.

$$\underbrace{\begin{bmatrix} 2 & 0 & -2 \\ -1 & 3 & 4 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 6 \end{bmatrix}}_{\vec{b}}$$

augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 4 \\ -1 & 3 & 4 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & -4 & -6 \\ 0 & 6 & 6 & 16 \end{array}\right]$$

pivots

We will return to this on Wednesday

5. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & -1 & 1 & -4 \\ 1 & 0 & 3 & -2 \\ 2 & 2 & 10 & 0 \end{bmatrix}$$

(a) Rephrase the following question in as many ways as you possibly can: *Is every vector* $\mathbf{b}$ *in* $\mathbb{R}^3$ *in the range of* T*?*

(b) What is the answer? Is every vector $\mathbf{b}$ in $\mathbb{R}^3$ in the range of T?