MATH 220
26 February 2025

BIG PICTURE:

We started this course studying linear systems, which arise in many (all?) areas of math, science, technology, etc.

$$\begin{cases} 2x_1 - 5x_2 + 3x_3 = 7 \\ x_1 + 2x_2 - 4x_3 = 2 \\ -5x_1 \quad\quad + 3x_3 = 6 \end{cases}$$

Linear systems are really linear combinations of vectors!

$$x_1 \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 6 \end{bmatrix}$$

We want to <u>really</u> understand linear combinations, especially in terms of <u>span</u> and <u>linear</u> <u>(in) dependence</u>.

Linear Algebra – Day 9

MATH 220

1. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ be a set of four mystery vectors in $\mathbb{R}^3$. In each case below, (i) decide whether the set is linearly independent or dependent, and (ii) decide whether the set spans all of $\mathbb{R}^3$.

(a) The matrix $\begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4\end{bmatrix}$ has RREF $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$.

$2\vec{u}_1 + 3\vec{u}_2 + 4\vec{u}_3 = \vec{u}_4$

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ are linearly dependent

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ are linearly independent

$\vec{u}_1, \vec{u}_2, \vec{u}_3$ span $\mathbb{R}^3$

(b) The matrix $\begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4\end{bmatrix}$ has RREF $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & ? \\ 0 & 1 & 3 & 0 & ? \\ 0 & 0 & 0 & 1 & ? \end{array}\right]$

dependent

span $\mathbb{R}^3$

(c) The matrix $\begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4\end{bmatrix}$ has RREF $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

dependent

don't span $\mathbb{R}^3$

2. In parts (a) – (c), circle the correct option for each statement.

$\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix}, \cdots \begin{bmatrix}0\\ \vdots\\0\\1\end{bmatrix}$

(a) $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{10}$ are ten vectors in $\mathbb{R}^9$.

The vectors must / cannot / might or might not span $\mathbb{R}^9$.

The vectors must / cannot / might or might not be linearly independent.

If these vectors span $\mathbb{R}^9$, then they must / cannot / might or might not be linearly independent.

(b) $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_8$ are eight vectors in $\mathbb{R}^9$.

The vectors must / cannot / might or might not span $\mathbb{R}^9$.

The vectors must / cannot / might or might not be linearly independent.

If these vectors are linearly independent, then they must / cannot / might or might not span $\mathbb{R}^9$.

(c) $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_9$ are nine vectors in $\mathbb{R}^9$.

The vectors must / cannot / might or might not span $\mathbb{R}^9$.

The vectors must / cannot / might or might not be linearly independent.

If these vectors span $\mathbb{R}^9$, then they must / cannot / might or might not be linearly independent.

3. Suppose A is a 4×4 matrix whose columns are linearly independent. Explain *why* the linear system whose augmented matrix is $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ has exactly one solution for all vectors $\mathbf{b} \in \mathbb{R}^4$.

Since the four columns of A are linearly independent, they span $\mathbb{R}^4$.

This means that any vector $\vec{b}$ in $\mathbb{R}^4$ can be written as a linear combination of the columns of A.

This linear combination is a solution to the system with augmented matrix $[A \mid b]$.

Since there are only 4 columns, the linear combination for each $\vec{b}$ in $\mathbb{R}^4$ is unique, and thus the solution to the system is unique.

4. Suppose A is a 4×4 matrix whose columns are linearly dependent. Explain *why* the linear system whose augmented matrix is $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ has no solution for some vector $\mathbf{b} \in \mathbb{R}^4$.

Since the columns of A are linearly dependent, the reduced form of A has at least one row of zeros.

Augment a column $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ to the reduced form of A and un-do the row operations to obtain a column $\vec{b}$ such that $[A \mid \vec{b}]$ has no solution.

5. In each case, determine whether $\mathbf{b}$ is in the span of the other vectors. If so, express $\mathbf{b}$ as a linear combination of the other vectors.

use Mathematica!

(a) $\mathbf{a}_1 = \begin{bmatrix} 4 \\ -3 \\ 4 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 & 7 \\ -3 & -6 & 1 \\ 4 & 2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so $\vec{b}$ is not in the span of $\vec{a}_1$ and $\vec{a}_2$.

(b) $\mathbf{a}_1 = \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 5 & 4 & -2 \\ -2 & 7 & 6 & -4 \\ 5 & -1 & 2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 3/13 \\ 0 & 1 & 0 & -4/13 \\ 0 & 0 & 1 & -3/13 \end{bmatrix}$$

So $\vec{b}$ is in $\text{Span}(\vec{a}_1, \vec{a}_2, \vec{a}_3)$, and in fact $\vec{b} = \frac{3}{13}\vec{a}_1 - \frac{4}{13}\vec{a}_2 - \frac{3}{13}\vec{a}_3$.

6. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent vectors in $\mathbb{R}^n$. Let matrix A have these vectors as its columns. What is the RREF of A?

Since the n vectors are linearly independent, the reduced form of A will have a pivot in each column. Therefore, the reduced form is:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

← That is, an $n \times n$ matrix with 1s along the diagonal and 0s elsewhere.

7. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent vectors in $\mathbb{R}^n$. Let matrix A have these vectors as its columns. What can you say about the RREF of A?

Since the columns are linearly dependent, the reduced form of A must have at least one row of zeros.

This is the big theorem from Chapters 1 and 2:

THEOREM 2.21

THE UNIFYING THEOREM – VERSION 1 Let $\mathcal{S} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbf{R}^n$, and let $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$. Then the following are equivalent:

either all true or all false

(a) $\mathcal{S}$ spans $\mathbf{R}^n$.

(b) $\mathcal{S}$ is linearly independent.

(c) $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$ in $\mathbf{R}^n$.