Linear Algebra – Day 8

MATH 220

1. Warm-up Group Chat:

(a) Is $\text{span}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\3\\4\\4\end{bmatrix}\right)$ the same as $\text{span}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right)$? Why or why not?

(b) Is $\text{span}\left(\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right)$ the same as $\text{span}\left(\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}4\\-2\\1\end{bmatrix}\right)$? Why or why not?

2. Consider the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ in $\mathbb{R}^n$.

(a) Suppose

$$\mathbf{v}_4 = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 5\mathbf{v}_3$$

Why does this mean $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are linearly dependent? Have a discussion with your group as to why.

(b) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are linearly dependent because

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 0\mathbf{v}_3 + 7\mathbf{v}_4 = \mathbf{0}$$

☞ Found a way to express $\mathbf{0}$ in a nontrivial way.

- Given only this information, explain how it is possible to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

- Given only this information, is it possible to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$?

3. Have a discussion with your table about the following questions:

(a) Are the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$ linearly independent or linearly dependent?

☞ If they are linearly dependent, find a dependence relation.

(b) Can you *generalize* what just happened?

☞ Hint: What happens if $\mathbf{0}$ is included among a list of vectors?

4. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly independent or linearly dependent?

☞ If they are linearly dependent, find a dependence relation.

(b) Do the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$?

☞ You shouldn't have to compute anything new to answer this.

(c) Let A be the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Does $A\mathbf{x} = \mathbf{b}$ have a unique solution for all $\mathbf{b}$ in $\mathbb{R}^3$?

5. For the following questions you always end up solving a system that has augmented matrix that looks something like $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ where you know A has n rows and m columns.

(a) What relationship between m and n guarantees the m columns of A **don't** span $\mathbb{R}^n$?

$m < n$ $\quad$ $m = n$ $\quad$ $m > n$

(b) What relationship between m and n guarantees the m columns of A **are not** linearly independent?

$m < n$ $\quad$ $m = n$ $\quad$ $m > n$

(c) Summary:

- The m columns of A could only span $\mathbb{R}^n$ if ______________.
- The m columns of A could only be linearly independent if ______________.
- The m columns of A could only span $\mathbb{R}^n$ AND be linearly independent if ______________.

6. Can you come up with an example of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (in $\mathbb{R}^2$) which are linearly *dependent*, but yet $\mathbf{v}_1$ is **not** a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$?