MATH 220
24 February 2025

RECALL: LINEAR DEPENDENCE & INDEPENDENCE

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent if the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}$$

has only the trivial solution.

$x_1 = x_2 = \cdots = x_k = 0$

If there are nontrivial solutions, then the vectors are linearly dependent.

IMPORTANT. A set of vectors is linearly dependent if some vector in the set is a linear combination of the other vectors.

A set of vectors is linearly independent if no vector in the set is a linear combination of the others

Linear Algebra – Day 8

MATH 220

1. Warm-up Group Chat:

linear combination
$v_4 = 0v_1 + 3v_2 + 4v_3$

(a) Is span $\left(\overset{v_1}{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}, \overset{v_2}{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}, \overset{v_3}{\begin{bmatrix}0\\0\\1\\1\end{bmatrix}}, \overset{v_4}{\begin{bmatrix}0\\3\\4\\4\end{bmatrix}} \right)$ the same as span $\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \right)$? Why or why not?

3-dimensional hyperplane in $\mathbb{R}^4$

$v_1 + 2v_2 = v_3$

(b) Is span $\underbrace{\left(\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix} \right)}_{\text{this span is a plane in } \mathbb{R}^3}$ the same as span $\left(\overset{v_1}{\begin{bmatrix}2\\0\\1\end{bmatrix}}, \overset{v_2}{\begin{bmatrix}1\\-1\\0\end{bmatrix}}, \overset{v_3}{\begin{bmatrix}4\\-2\\1\end{bmatrix}} \right)$? Why or why not?

2. Consider the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ in $\mathbb{R}^n$.

(a) Suppose

$$\mathbf{v}_4 = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 5\mathbf{v}_3$$

Why does this mean $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are linearly dependent? Have a discussion with your group as to why.

$2\vec{v}_1 + 3\vec{v}_2 - 5\vec{v}_3 - \vec{v}_4 = \vec{0}$

gives a nontrivial solution to $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$

(b) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are linearly dependent because

☞ Found a way to express **0** in a nontrivial way.

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 0\mathbf{v}_3 + 7\mathbf{v}_4 = \mathbf{0}$$

- Given only this information, explain how it is possible to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

  $\vec{v}_4 = \frac{-3}{7}\vec{v}_1 + \frac{5}{7}\vec{v}_2 + 0\vec{v}_3$

- Given only this information, is it possible to express $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$? $\vec{0} = 0\vec{v}_3 = -3\vec{v}_1 + 5\vec{v}_2 - 7\vec{v}_4$ ← does not express $\vec{v}_3$ as a linear comb.

  No.

  However, it is still possible that $\vec{v}_3$ could be a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_4$. For example, it could be that $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$.

3. Have a discussion with your table about the following questions:

(a) Are the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\0\end{bmatrix}$ linearly independent or linearly dependent?

☞ If they are linearly dependent, find a dependence relation.

$\begin{bmatrix}0\\0\\0\end{bmatrix} = 0\begin{bmatrix}1\\0\\1\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix}$

$0\begin{bmatrix}1\\0\\1\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(b) Can you *generalize* what just happened?

☞ Hint: What happens if **0** is included among a list of vectors?

If $\vec{0}$ is among a set of vectors, then the set of vectors is linearly dependent.

4. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -4 & 2 \\ 1 & -2 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

(a) Are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly independent or linearly dependent?

If they are linearly dependent, find a dependence relation.

$$2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Unifying Theorem

(b) Do the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$?

You shouldn't have to compute anything new to answer this.

No, since we don't have 3 linearly independent vectors

(c) Let A be the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Does $A\mathbf{x} = \mathbf{b}$ have a unique solution for all $\mathbf{b}$ in $\mathbb{R}^3$?

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 2 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

— class ended here —

5. For the following questions you always end up solving a system that has augmented matrix that looks something like $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ where you know A has n rows and m columns.

(a) What relationship between m and n guarantees the m columns of A **don't** span $\mathbb{R}^n$?

$m < n \qquad m = n \qquad m > n$

(b) What relationship between m and n guarantees the m columns of A **are not** linearly independent?

$m < n \qquad m = n \qquad m > n$

(c) Summary:

- The m columns of A could only span $\mathbb{R}^n$ if ______________.
- The m columns of A could only be linearly independent if ______________.
- The m columns of A could only span $\mathbb{R}^n$ AND be linearly independent if ______________.

6. Can you come up with an example of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (in $\mathbb{R}^2$) which are linearly *dependent*, but yet $\mathbf{v}_1$ is **not** a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$?