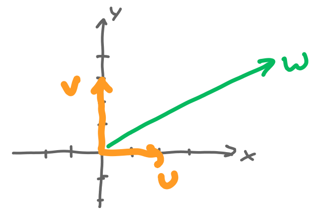
MATH 220
21 February 2025

**SPAN.** span$(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$ is the collection of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$

$\vec{u}$ and $\vec{v}$ span the plane $\mathbb{R}^2$

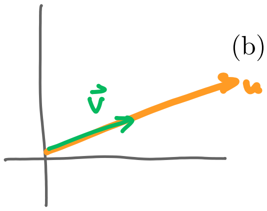
$\leftarrow$ verb




**4.** (a) Let $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Explain why $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$.

$\vec{w} = x_1\vec{u} + x_2\vec{v}$ has a solution

☞ When we say $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$, what we mean is EVERY vector in $\mathbb{R}^2$ is in $\text{span}(\mathbf{u}, \mathbf{v})$.




(b) Now let $\mathbf{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

i. Explain why $\text{span}(\mathbf{u}, \mathbf{v}) \neq \mathbb{R}^2$.

ii. What geometric "shape" is formed by $\text{span}(\mathbf{u}, \mathbf{v})$? — a line

☞ That is, $\text{span}(\mathbf{u}, \mathbf{v})$ is *not* ALL of $\mathbb{R}^2$.

(c) Let $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Is $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$?

Not multiples of each other, so they span $\mathbb{R}^2$

(d) You have now seen two different situations: one in which $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$ and another in which $\mathbb{R}^2 \neq \text{span}(\mathbf{u}, \mathbf{v})$.

**Group chat/conjecture:** What must be true about $\mathbf{u}, \mathbf{v}$ in order for $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$?

$\vec{u}$ and $\vec{v}$ are <u>not</u> on the same line

☞ What I mean is, what can you say about their relationship to one another?

**5.** Now that we have started working with "span" a little bit, let's test our intuition:

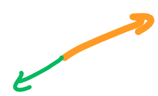
☞ You cant draw in $\mathbb{R}^4$ but just have an imagination!

(a) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$? line

(b) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$? plane

☞ Hint: look very carefully at how the specific vectors are or are not related.

(c) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 0 \\ 0 \end{bmatrix}\right)$? line



☞ Hint: look very carefully at how the specific vectors are or are not related.

(d) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$? plane

☞ Hint: look very carefully at how the specific vectors are or are not related.

(e) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$? 3-dimensional subspace

☞ Hint: Do I really need to type it AGAIN?

all vectors of the form $\begin{bmatrix} a \\ a \\ b \\ c \end{bmatrix}$

**6.** In $\mathbb{R}^3$, which vector(s), if any, are in $\text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$?

only $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

# New notation for a linear system

example: system
$$\begin{cases} x_1 + 4x_2 = 7 \\ 2x_1 + 5x_2 = 8 \\ 3x_1 + 6x_2 = 9 \end{cases}$$

vector equation: $x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

matrix/vector equation: $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

form: $A \vec{x} = \vec{b}$

coefficient matrix

vector of unknowns $x_1, x_2$

vector of known constants

---

# NEW DEFINITION:

If the vector equation
$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}$$
has nontrivial solutions, then the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are **linearly dependent.**

If this equation has only the trivial solution, then the vectors are **linearly independent.**

# Linear Algebra – Day 7

MATH 220

$-1\begin{bmatrix}1\\2\\3\end{bmatrix}+2\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}7\\8\\9\end{bmatrix}$

$\begin{bmatrix}7\\8\\9\end{bmatrix}$ depends on $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}$

**1.** (a) **Erez:** I just noticed that the RREF of $\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$ is $\left[\begin{array}{cc|c}1&0&-1\\0&1&2\\0&0&0\end{array}\right]$.

**Cleo:** That means that $x_1\begin{bmatrix}1\\2\\3\end{bmatrix}+x_2\begin{bmatrix}4\\5\\6\end{bmatrix}+x_3\begin{bmatrix}7\\8\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ has a nontrivial solution!

not $x_1=x_2=x_3=0$

$\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\\0&0&0&0\end{array}\right]$

**Group chat:** Is Cleo correct? How many solutions does this equation have? infinitely many

$-1\begin{bmatrix}1\\2\\3\end{bmatrix}+2\begin{bmatrix}4\\5\\6\end{bmatrix}-1\begin{bmatrix}7\\8\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ ← multiply this by whatever you like

(b) **Maura:** That means you can write $\begin{bmatrix}7\\8\\9\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\5\\6\end{bmatrix}$!

**Group chat:** Is Maura correct? How do you know?

Yes

(c) **Milo:** That means that $\begin{bmatrix}7\\8\\9\end{bmatrix}$ is in span $\left(\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right)$!

$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$ are linearly dependent

**Group chat:** Is Milo correct? How do you know?

Yes, since span is the set of all linear combinations.

**2.** Let $\mathbf{u}_1=\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\mathbf{u}_2=\begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\mathbf{u}_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\mathbf{u}_4=\begin{bmatrix}5\\4\\9\end{bmatrix}$

$$\begin{array}{ccc} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{array}$$
$$\left[\begin{array}{ccc|c}1&-1&1&0\\2&1&1&0\\3&0&1&0\end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

(a) Does $x_1\mathbf{u}_1+x_2\mathbf{u}_2+x_3\mathbf{u}_3=\mathbf{0}$ have a non-trivial solution? If so, how many?

No.

Solution: $x_1=0$, $x_2=0$, $x_3=0$

(b) Is the set $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ linearly independent or dependent?

class ended here

(c) Is $\mathbf{u}_1\in\operatorname{span}(\mathbf{u}_2,\mathbf{u}_3)$?

(d) Show that $\mathbf{u}_4\in\operatorname{span}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)$ and that there is only one way to write $\mathbf{u}_4$ as a linear combination of the others.

**3.** Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{u}_4 = \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix}$

(a) Does $x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = \mathbf{0}$ have a non-trivial solution? If so, how many?

(b) Is the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ linearly independent or dependent?

(c) Show that $\mathbf{u}_4 \in \operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ and that there are infinitely many ways to write $\mathbf{u}_4$ as a linear combination of the others.

(d) Find two different ways to write $\mathbf{u}_4$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

**4.** Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a set of three mystery vectors in $\mathbb{R}^3$. In each case below, (i) decide whether the set is linearly independent or dependent, and (ii) decide whether the set spans all of $\mathbb{R}^3$.

(a) The matrix $\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$ has RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) The matrix $\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$ has RREF $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.