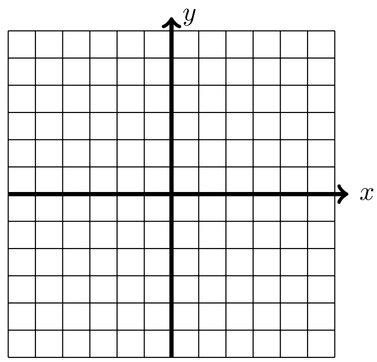
Linear Algebra – Day 6

MATH 220

1. (a) On the grid below, draw $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and the line created by all the multiples of $\mathbf{u}$.

(b) On the grid below, draw $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and the line created by all the multiples of $\mathbf{v}$.



(c) The two lines you have drawn should break the xy-plane into four regions.

(i) If a new vector $\mathbf{w}$ is created by adding a *positive* multiple of $\mathbf{u}$ with a *positive* multiple of $\mathbf{v}$, in which region is $\mathbf{w}$?

(ii) If $\mathbf{w}$ is created by adding a *negative* multiple of $\mathbf{u}$ with a *positive* multiple of $\mathbf{v}$, in which region is $\mathbf{w}$?

(iii) If $\mathbf{w}$ is created by adding a *positive* multiple of $\mathbf{u}$ with a *negative* multiple of $\mathbf{v}$, in which region is $\mathbf{w}$?

(iv) If $\mathbf{w}$ is created by adding a *negative* multiple of $\mathbf{u}$ with a *negative* multiple of $\mathbf{v}$, in which region is $\mathbf{w}$?

☞ Remember when you add two vectors, the result is the diagonal of the parallelogram they create.

(d) Is the zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?

(e) Is there any vector in $\mathbb{R}^2$ that is *not* a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

2. Group Discussion: Describe which of the vectors in $\mathbb{R}^3$ are in span $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

☞ New notation: $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3. In $\mathbb{R}^3$, let $\quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}.$

(a) Is the vector $\mathbf{b}_1$ in span($\mathbf{u}$, $\mathbf{v}$)? What about the vector $\mathbf{b}_2$?

☞ Rephrase the question into a question about linear combinations. KEEP REPHRASING the question!

☞ Suggestion: Mathematica

(b) Is the zero vector, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, contained in span($\mathbf{u}, \mathbf{v}$)? Explain.

4. (a) Let $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Explain why $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$.

☞ When we say $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$, what we mean is EVERY vector in $\mathbb{R}^2$ is in $\text{span}(\mathbf{u}, \mathbf{v})$.

(b) Now let $\mathbf{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

i. Explain why $\text{span}(\mathbf{u}, \mathbf{v}) \neq \mathbb{R}^2$.

ii. What geometric "shape" is formed by $\text{span}(\mathbf{u}, \mathbf{v})$?

☞ That is, $\text{span}(\mathbf{u}, \mathbf{v})$ is *not* ALL of $\mathbb{R}^2$.

(c) Let $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Is $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$?

(d) You have now seen two different situations: one in which $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$ and another in which $\mathbb{R}^2 \neq \text{span}(\mathbf{u}, \mathbf{v})$.

Group chat/conjecture: What must be true about $\mathbf{u}, \mathbf{v}$ in order for $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$?

☞ What I mean is, what can you say about their relationship to one another?

5. Now that we have started working with "span" a little bit, let's test our intuition:

☞ You cant draw in $\mathbb{R}^4$ but just have an imagination!

(a) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$?

(b) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$?

☞ Hint: look very carefully at how the specific vectors are or are not related.

(c) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 0 \\ 0 \end{bmatrix}\right)$?

☞ Hint: look very carefully at how the specific vectors are or are not related.

(d) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$?

☞ Hint: look very carefully at how the specific vectors are or are not related.

(e) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$?

☞ Hint: Do I really need to type it AGAIN?

6. In $\mathbb{R}^3$, which vector(s), if any, are in $\text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$?