MATH 220
19 February 2025

VECTORS:
Vectors can be used to express solutions to linear systems.

example:
$$\begin{cases} 2x_1 - x_2 + x_3 + x_4 = 1 \\ 4x_1 - x_2 + 3x_3 + 5x_4 = 3 \\ 2x_1 \quad\quad + 2x_3 + 4x_4 = 2 \end{cases}$$

has solutions:
$$x_1 = 1 - x_3 - 2x_4$$
$$x_2 = 1 - x_3 - 3x_4$$
x_3, x_4 are free

We can put the solutions into a solution vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - x_3 - 2x_4 \\ 1 - x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{constants}} + x_3 \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{coeffs of } x_3} + x_4 \underbrace{\begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\text{coeffs of } x_4}$$

IMPORTANT SKILL: The ability to rephrase a problem or a situation

QUESTION: Is $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$?

Rephrase using the definition of linear combination:

Is it possible to find x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} ?$$

Rephrase using knowledge of vector algebra:

Is there a solution to the system
$$\begin{cases} 1x_1 + 2x_2 + 3x_3 = 2 \\ 3x_1 + 1x_2 - 1x_3 = 0 \\ 4x_1 + 3x_2 - 2x_3 = 2 \end{cases} \; ?$$

Rephrase using an augmented matrix:

Does the augmented matrix
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 3 & 1 & -1 & 0 \\ 4 & 3 & -2 & 2 \end{array}\right]$$
have a reduced form that avoids a pivot in the rightmost column?

ANSWER: Use Mathematica to reduce the matrix:
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2/5 \\ 0 & 1 & 0 & 6/5 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Solution: $x_1 = \frac{-2}{5}, \quad x_2 = \frac{6}{5}, \quad x_3 = 0$

So: $$-\frac{2}{5}\begin{bmatrix}1\\3\\4\end{bmatrix} + \frac{6}{5}\begin{bmatrix}2\\1\\3\end{bmatrix} + 0\begin{bmatrix}3\\-1\\-2\end{bmatrix} = \begin{bmatrix}2\\0\\2\end{bmatrix}$$

If you have vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$, then each vector in $\mathbb{R}^n$ either <u>is</u> or <u>is not</u> a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

<u>example:</u> $\begin{bmatrix}1\\1\\0\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$

$\begin{bmatrix}0\\0\\1\end{bmatrix}$ is <u>not</u> a linear combination of $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$

Definition: The collection of all vectors that are linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is called the **span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$.

notation: $\operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$

So, another way of rephrasing our initial question is:

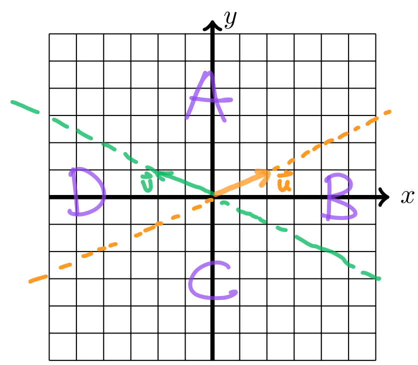
Is $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ in $\text{span}\left(\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} \right)$?

Linear Algebra – Day 6

MATH 220

1. (a) On the grid below, draw $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and the line created by all the multiples of $\mathbf{u}$.

(b) On the grid below, draw $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and the line created by all the multiples of $\mathbf{v}$.



(c) The two lines you have drawn should break the xy-plane into four regions.

(i) If a new vector $\mathbf{w}$ is created by adding a *positive* multiple of $\mathbf{u}$ with a *positive* multiple of $\mathbf{v}$, in which region is $\mathbf{w}$?

A

(ii) If $\mathbf{w}$ is created by adding a *negative* multiple of $\mathbf{u}$ with a *positive* multiple of $\mathbf{v}$, in which region is $\mathbf{w}$?

D

(iii) If $\mathbf{w}$ is created by adding a *positive* multiple of $\mathbf{u}$ with a *negative* multiple of $\mathbf{v}$, in which region is $\mathbf{w}$?

B

(iv) If $\mathbf{w}$ is created by adding a *negative* multiple of $\mathbf{u}$ with a *negative* multiple of $\mathbf{v}$, in which region is $\mathbf{w}$?

C

Remember when you add two vectors, the result is the diagonal of the parallelogram they create.

(d) Is the zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$? $0\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Yes!

(e) Is there any vector in $\mathbb{R}^2$ that is *not* a linear combination of $\mathbf{u}$ and $\mathbf{v}$. No.

$\text{Span}(\vec{u}, \vec{v}) = \mathbb{R}^2$

2. Group Discussion: Describe which of the vectors in $\mathbb{R}^3$ are in span $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \mathbb{R}^3$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

New notation: $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3. In $\mathbb{R}^3$, let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}$.

(a) Is the vector $\mathbf{b}_1$ in span($\mathbf{u}$, $\mathbf{v}$)? What about the vector $\mathbf{b}_2$?

yes

Solve: $x_1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{\text{matrix}} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 1 & -1 & -4 \\ 2 & 6 & 2 \end{array}\right]$

$\downarrow$ reduce

$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow$ solution

Rephrase the question into a question about linear combinations. KEEP REPHRASING the question!

Suggestion: Mathematica

(b) Is the zero vector, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, contained in span($\mathbf{u}, \mathbf{v}$)? Explain.

Yes! $0\vec{u} + 0\vec{v} = 0$

to be continued...

4. (a) Let $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Explain why $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$.

When we say $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$, what we mean is EVERY vector in $\mathbb{R}^2$ is in $\text{span}(\mathbf{u}, \mathbf{v})$.

(b) Now let $\mathbf{u} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

i. Explain why $\text{span}(\mathbf{u}, \mathbf{v}) \neq \mathbb{R}^2$.

ii. What geometric "shape" is formed by $\text{span}(\mathbf{u}, \mathbf{v})$?

That is, $\text{span}(\mathbf{u}, \mathbf{v})$ is *not* ALL of $\mathbb{R}^2$.

(c) Let $\mathbf{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Is $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$?

(d) You have now seen two different situations: one in which $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$ and another in which $\mathbb{R}^2 \neq \text{span}(\mathbf{u}, \mathbf{v})$.

Group chat/conjecture: What must be true about $\mathbf{u}, \mathbf{v}$ in order for $\mathbb{R}^2 = \text{span}(\mathbf{u}, \mathbf{v})$?

What I mean is, what can you say about their relationship to one another?

5. Now that we have started working with "span" a little bit, let's test our intuition:

You cant draw in $\mathbb{R}^4$ but just have an imagination!

(a) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right)$?

(b) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$?

Hint: look very carefully at how the specific vectors are or are not related.

(c) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \\ 0 \\ 0 \end{bmatrix}\right)$?

Hint: look very carefully at how the specific vectors are or are not related.

(d) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$?

Hint: look very carefully at how the specific vectors are or are not related.

(e) What geometric "shape" is formed by $\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$?

Hint: Do I really need to type it AGAIN?

6. In $\mathbb{R}^3$, which vector(s), if any, are in $\text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$?