MATH 220

17 February 2025

VECTORS: A vector is an ordered list of n numbers.

examples:

$n=2$

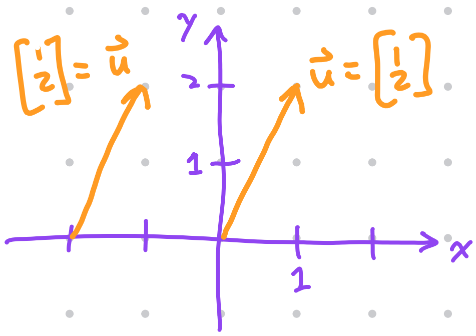
$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -5 \\ 0.2 \end{bmatrix}$$

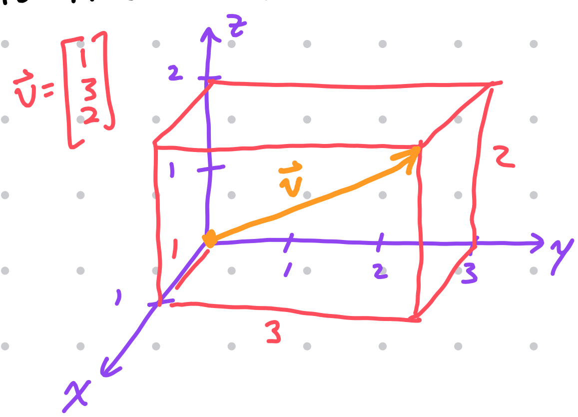
$n=3$

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Vectors can be visualized as arrows

The numbers tell you how far to move in each direction.



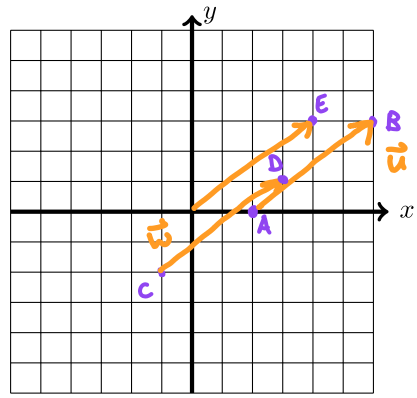


NOTATION: $\mathbb{R}^n$ is the collection of <u>all</u> vectors with n components

Linear Algebra – Day 5

MATH 220

1. Let $A = (2, 0)$ and $B = (6, 3)$. Draw the vector from A to B (which we will call $\mathbf{u}$).



$\overrightarrow{AB} = \vec{u}$

(a) Suppose $C = (-1, -2)$ and the vector from C to D (which we will call $\mathbf{w}$) has the same length and direction as the vector $\mathbf{u}$. Find the coordinates of point D.

$D = (3, 1)$

(b) Sketch vector $\mathbf{w}$ on the grid above. Geometrically, how are the vector $\mathbf{u}$ and the vector $\mathbf{w}$ related?

parallel and same length

(c) Let $O = (0, 0)$ denote the origin. Find point E such that the vector from O to E *equals* the vector $\mathbf{u}$.

☞ Unless we have a reason not to, we almost always draw vectors emanating from the origin.

2. Consider the vectors $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a) What do you think $\mathbf{u} + \mathbf{v}$ should equal?

$\vec{u} + \vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

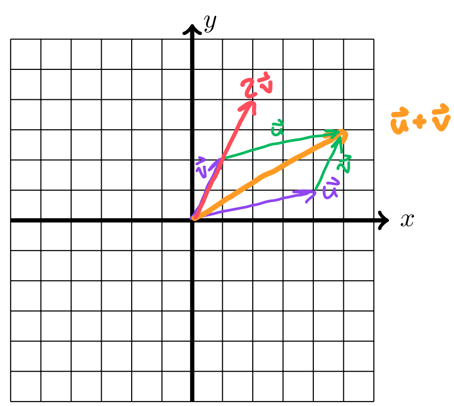
(b) What do you think $2\mathbf{v}$ should equal?

$2\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(c) Sketch $\mathbf{u}$, $\mathbf{v}$, and (your guesses for) $\mathbf{u} + \mathbf{v}$ and $2\mathbf{v}$ on the grid below (all emanating from the origin). In the sketch, how are these vectors related to each other?

(d) Use your sketch to explain why $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

☞ This is "obvious" algebraically. Why?



VECTOR ALGEBRA:

add: $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix} = \begin{bmatrix} c+a \\ d+b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$

multiply by a $\underbrace{\text{scalar}}_{\text{number}}$: $k \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} k \cdot a \\ k \cdot b \end{bmatrix}$

LINEAR COMBINATION:

of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ in $\mathbb{R}^n$ is any vector of the form

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k$$

for some constants (scalars) $c_1, c_2, \dots, c_k$

linear combination of vectors

3. Is it possible to find numbers x_1, x_2, x_3 so that $x_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + x_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + x_3 \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\0\\2\end{bmatrix}$?

☜ All three positions on each side of the equation must be equal.

$$\begin{bmatrix}x_1 + x_2 + x_3\\ x_2 + x_3\\ x_3\end{bmatrix} = \begin{bmatrix}2\\0\\2\end{bmatrix}$$

$x_3 = 2$

$x_2 + x_3 = 0 \Rightarrow x_2 + 2 = 0$

$\Rightarrow x_2 = -2$

$x_1 + x_2 + x_3 = 2 \Rightarrow x_1 + (\cancel{-2}) + (\cancel{2}) = 2$ so $x_1 = 2$

rephrase: Is $\begin{bmatrix}2\\0\\2\end{bmatrix}$ a linear combination of $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$?

4. (a) Can you find numbers x_1 and x_2 such that $x_1 \begin{bmatrix}1\\0\\-1\end{bmatrix} + x_2 \begin{bmatrix}2\\0\\3\end{bmatrix} = \begin{bmatrix}2\\1\\-1\end{bmatrix}$?

No, since $0x_1 + 0x_2 \neq 1$

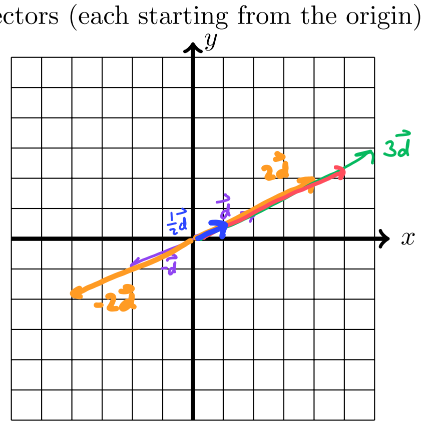
(b) Can you find numbers x_1 and x_2 such that $x_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + x_2 \begin{bmatrix}1\\5\\3\end{bmatrix} = \begin{bmatrix}1\\-1\\3\end{bmatrix}$?

$x_1 = 2$

$x_2 = -1$

5. Suppose we have the vector $\mathbf{d} = \begin{bmatrix}2\\1\end{bmatrix}$.

(a) Draw each the following vectors (each starting from the origin): $\mathbf{d}, 2\mathbf{d}, 3\mathbf{d}$, and $\frac{5}{2}\mathbf{d}$.



$\frac{5}{2}\vec{d} = \begin{bmatrix}5\\5/2\end{bmatrix}$

all multiples $k \cdot \vec{d}$ lie along a line (linear combinations of the single vector $\vec{d}$)

(b) Suppose t is some random number that is larger than 1. What does the vector $t\mathbf{d}$ look like compared to $\mathbf{d}$?

☜ That is, how is $t\mathbf{d}$ related to $\mathbf{d}$ when you draw them both?

(c) Draw the vector $\frac{1}{2}\mathbf{d}$ above. Suppose t is some random number between 0 and 1. What does the vector $t\mathbf{d}$ look like compared to $\mathbf{d}$?

(d) On the same grid above, draw the vectors $-\mathbf{d}, -2\mathbf{d}, -3\mathbf{d}, -\frac{5}{2}\mathbf{d}$. If t is some random negative number, describe what the vector $t\mathbf{d}$ looks like.

(e) If you look at *all* the vectors of the form $t\mathbf{d}$ together at the same time, what geometric shape do they create (if any)?