MATH 220

12 February 2025

A linear system is in **ECHELON FORM** if it is organized in a stairstep form: the leading variables move to the right as you read down the equations.

EXAMPLE:

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 1 \\ \qquad\qquad x_3 + x_4 = 2 \end{cases}$$

$x_1 = 1 - x_2 - 2x_3 - x_4$

$= 1 - x_2 - 2(2 - x_4) - x_4$ $+2x_4$ $-1x_4$

$\boxed{x_1 = -3 - x_2 + x_4}$

$\boxed{x_3 = 2 - x_4}$

Solution set:

$x_1 = -3 - s_1 + s_2$

$x_2 = s_1$

$x_3 = 2 - s_2$

$x_4 = s_2$

where s_1 and s_2 are real numbers

4. Here are three systems of equations:

👈 You'll need one free parameter/variable for (a) and (b), and two for (c).

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + x_3 &= 2 \end{aligned} \qquad \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_2 - x_3 \quad\; &= 2 \\ x_4 &= 3 \end{aligned} \qquad \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_3 + x_4 &= 3 \end{aligned}$$

(a) **Group discussion:** Why are each of these easier to solve than the system at the top of page #1?

(b) **Group discussion:** Try actually finding the solutions to each system.

(c) **Ava:** Milo! I can tell JUST BY LOOKING at each of these systems in #4 *exactly* how many free variables there will be!! WOW!!

Group chat: Can you figure why Ava knows right away how many free variables there are going to be?

From last time:

5. Below are steps that Jason used to solve a system. To the *right* of each step, write down the one thing Jason did, if you can figure it out!

START:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 4x_2 + 8x_3 &= 10 \\ x_1 + 3x_2 + 4x_3 &= 6 \end{aligned}$$

STEP 1: Subtract row 1 from row 3

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 4x_2 + 8x_3 &= 10 \\ x_2 + x_3 &= 2 \end{aligned}$$

$-2x_1 - 4x_2 - 6x_3 = -8$

STEP 2: add -2 times row 1 to row 2

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_3 &= 2 \\ x_2 + x_3 &= 2 \end{aligned}$$

STEP 3: Swap rows 2 and 3

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + x_3 &= 2 \\ 2x_3 &= 2 \end{aligned}$$

STEP 4: divide row 3 by 2

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + x_3 &= 2 \\ x_3 &= 1 \end{aligned}$$

STEP 5: NOW back-substitution works! What's the solution to the system?

$x_3 = 1$

$x_2 = 2 - x_3 = 2 - 1 = 1$

$x_1 = 4 - 2x_2 - 3x_3 = 4 - 2(1) - 3(1) = 4 - 2 - 3 = -1$

ELEMENTARY (ROW) OPERATIONS

1. Swap two equations.
2. Multiply an equation by a nonzero (number) constant.
3. Add a multiple of one equation to another equation.

These 3 operations are enough to get a system into echelon form!

Linear Algebra – Day 3

MATH 220

1. Group chat: Which row operation are performed between each step below? Also write the augmented matrix for the system at each step.

START:

$$\begin{aligned} 2x_2 + 4x_3 &= 2 \\ x_1 + 2x_2 + 3x_3 &= 1 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

=

$$\left[\begin{array}{ccc|c} 0 & 2 & 4 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right] \leftarrow \text{equation 1}$$

x_1 x_2 x_3 coeffs on right side

STEP 1: $R_1 \leftrightarrow R_2$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_2 + 4x_3 &= 2 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

STEP 2: $\frac{1}{2}R_2 \to R_2$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ x_2 + 2x_3 &= 1 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

STEP 3: $-R_2 + R_3 \to R_3$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ x_2 + 2x_3 &= 1 \\ 0 &= 0 \end{aligned}$$

pivot entries

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Felix: Alright Maura! The system is in echelon form. I'm ready to solve the system!

Maura: But I *LOVE* row operations! Can I just do one more, please? Here is STEP 4:

$-2R_2 + R_1 \to R_1$

$$\begin{aligned} x_1 \quad - x_3 &= -1 \\ x_2 + 2x_3 &= 1 \\ 0 &= 0 \end{aligned}$$

Felix: That was really useful! I find it *much* easier to solve the system now.

Group Chat: Why is it easier to solve the system after Maura's row operation?

2. Simon: I just put three different systems into an augmented matrix. Then, I did some row operations on all three. Here's what I got when I was all done.

(a) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 0 & 9 \end{array}\right]$

$-x_2 = 5 - 3x_3$

no solution!

Naila: Look, all three matrices are all in echelon form!

Group Discussion: Is Naila right? Are all three in echelon form?

Simon: This makes it easy to find the solutions to each system. Try it!

Group Discussion: What are the solutions to each system?

$x_1 = 3$
$x_2 = 1$
$x_3 = 2$

$x_1 = 2 - x_2 + x_3 = 2 - (-5+3s) + s = 7 - 2s$
$x_2 = -5 + 3s$
$x_3 = s$

3. Simon: Hey Maura, I need you to to help me with some systems.

Naila: It looks like you already put each system into an augmented matrix **and** did some row operations!

Simon: Yes. I don't care what the exact solution is, but I need to know, very quickly, *how many* solutions each system has.

(a) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$ 1 solution $5x_3 = 10$

(b) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 0 & 10 \end{array}\right]$ $0x_3 = 10$ no solution

(c) $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ∞ solutions

(d) $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right]$ 1 solution

(e) $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $3x_2 = 0$ 1 solution

(f) $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \end{array}\right]$ ∞ solutions

CLASS ENDED HERE

4. Group Discussion/Summary: Suppose you have an augmented matrix in echelon form.

Here are answers to #4 and 5:

(a) How can you tell when the corresponding system is inconsistent? Explain.

☞ i.e., has 0 solutions.

A row of the form $0 = c$, where c is a nonzero constant, indicates an inconsistent system.

(b) How can you tell when the corresponding system is consistent? Explain.

☞ i.e., has either 1 or ∞-many solutions.

If each nonzero row has a leading (nonzero) coeffient to the left of the vertical bar, then the system is consistent

(c) How can you tell when the corresponding system has *exactly* one solution? Explain.

If each column left of the vertical bar contains the pivot entry in some row, then there are no free variables and the system has exactly one solution.

(d) How can you tell when the corresponding system has ∞-many solutions? Explain.

If some column left of the vertical bar contains no pivot entry, then there is a free variable and the system has ∞-many solutions.

5. Suppose $\star$ represents the presence of a *nonzero* number. For each of the following augmented matrices, how many solutions will the corresponding system have?

☞ HINT: Think about how back-substitution might go.

$\left[\begin{array}{ccc|c} \star & \star & \star & \star \\ 0 & \star & \star & \star \end{array}\right]$ ∞-many

$\left[\begin{array}{ccc|c} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 \end{array}\right]$ ∞-many

$\left[\begin{array}{ccc|c} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & 0 & \star \end{array}\right]$ none

$\left[\begin{array}{ccc|c} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & 0 \end{array}\right]$ exactly one

$\left[\begin{array}{cc|c} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & 0 \end{array}\right]$ exactly one