MATH 220

10 February 2025

SYSTEMS OF LINEAR EQUATIONS

simple case: 2 variables

examples:

(a) $\begin{cases} 3x - 2y = 4 \\ 2x + 2y = 6 \end{cases}$ (b) $\begin{cases} x - 2y = 2 \\ -2x + 4y = 3 \end{cases}$ (c) $\begin{cases} -2x + 4y = 2 \\ x - 2y = -1 \end{cases}$

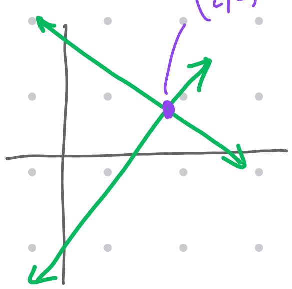
What do the graphs of these equations look like? LINES!

$y = \frac{3}{2}x - 2$

$y = -x + 3$

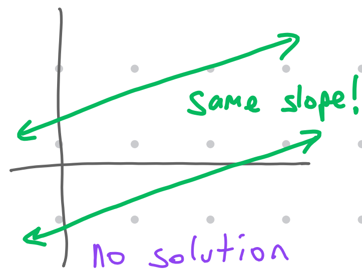
one solution

(2,1)



$y = \frac{1}{2}x - 1$

$y = \frac{1}{2}x + \frac{3}{4}$

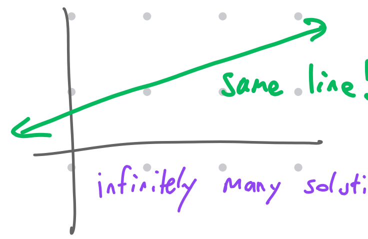


no solution

inconsistent

$y = \frac{1}{2}x + \frac{1}{2}$

$y = \frac{1}{2}x + \frac{1}{2}$



infinitely many solutions!

free variable → x: any number

$y = \frac{1}{2}x + \frac{1}{2}$

New terminology:

consistent: linear system with at least one solution

inconsistent: linear system with no solution

Linear Algebra – Day 2

MATH 220

EVENTUAL GOAL: We want to *eventually* (not now) solve the system

$$\begin{aligned} x + 2y + 3z &= 4 \\ 2x + 4y + 8z &= 10 \\ x + 3y + 4z &= 6 \end{aligned}$$

Quick Group Discussion: If you were going to try to solve this system right NOW, what strategies might you try?

DO NOT actually do it.

1. If we were to graph each of the three equations above, they would each be a *plane* in 3 dimensions.

Milo: Hey Ava. I think it's really cool that in 2 dimensions, two lines can either be parallel, intersect in exactly one point, or be the same line.

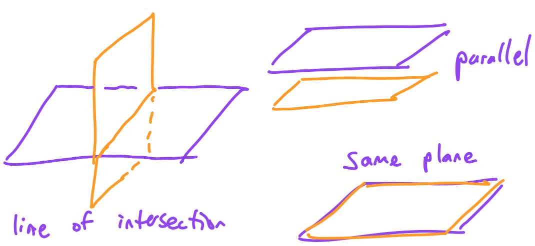
Ava: That is cool! I wonder what happens with two planes in 3 dimensions.

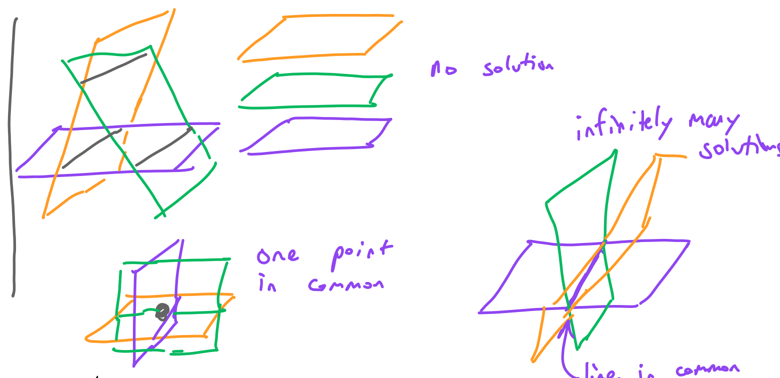
Group chat: Investigate! How could two planes interact in 3 dimensions?

Try drawing or using your hands or...

Milo: That was cool. Now what about THREE planes! How could THREE planes interact?

Group chat: Well? What do you think?





2. Find all the solutions for the following system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + x_3 &= 2 \\ x_3 &= 1 \end{aligned}$$

$x_2 + 1 = 2$

$x_1 + 2(1) + 3(1) = 4$

$x_1 + 5 = 4$

solution

$$\left\{ \begin{aligned} x_3 &= 1 \\ x_2 &= 1 \\ x_1 &= -1 \end{aligned} \right.$$

- This system is (circle one) consistent / inconsistent.
- The solution set (circle one) has a free variable / doesn't have a free variable.

3. Ava: Hello, Jason. Did you notice that the system at the top of the page and the system in #2 both involve three equations and three unknown variables?

Jason: Wassup, Ava? So, this means the two systems are equally difficult to solve.

Group discussion: What is your response to Jason?

Be nice.

4. Here are three systems of equations:

free variable

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 4 \\
x_2 + x_3 &= 2
\end{aligned}
\qquad
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 1 \\
x_2 - x_3 \qquad &= 2 \\
x_4 &= 3
\end{aligned}
\qquad
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 1 \\
x_3 + x_4 &= 3
\end{aligned}
$$

x_3 is a free variable

free variables

You'll need one free parameter/variable for (a) and (b), and two for (c).

(a) **Group discussion:** Why are each of these easier to solve than the system at the top of page #1?

(b) **Group discussion:** Try actually finding the solutions to each system.

$x_1 = -x_3$

$x_2 = 2 - x_3$

or

$x_1 = 4 - 2x_2 - 3x_3 = 4 - 2(2-s) - 3(s) = -s$

$4 - 4 + 2s - 3s$

$x_2 = 2 - s$

$x_3 = s$ ← free variable

$x_4 = 3$

$x_3 = t$

$x_2 = 2 + t$

$x_1 = 1 - x_2 - x_3 - x_4 = 1 - (2+t) - t - 3 = -4 - 2t$

$x_4 = s$

$x_3 = 3 - s$

$x_2 = t$

$x_1 = 1 - x_2 - x_3 - x_4 = 1 - t - (3-s) - s$
$= -2 - t$

(c) **Ava:** Milo! I can tell JUST BY LOOKING at each of these systems in #4 *exactly* how many free variables there will be!! WOW!!

Group chat: Can you figure why Ava knows right away how many free variables there are going to be?

5. Below are steps that Jason used to solve a system. To the *right* of each step, write down the one thing Jason did, if you can figure it out!

START:

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 4 \\
2x_1 + 4x_2 + 8x_3 &= 10 \\
x_1 + 3x_2 + 4x_3 &= 6
\end{aligned}
$$

STEP 1:

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 4 \\
2x_1 + 4x_2 + 8x_3 &= 10 \\
x_2 + x_3 &= 2
\end{aligned}
$$

STEP 2:

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 4 \\
2x_3 &= 2 \\
x_2 + x_3 &= 2
\end{aligned}
$$

STEP 3:

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 4 \\
x_2 + x_3 &= 2 \\
2x_3 &= 2
\end{aligned}
$$

STEP 4:

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 4 \\
x_2 + x_3 &= 2 \\
x_3 &= 1
\end{aligned}
$$

triangular form

STEP 5: NOW back-substitution works! What's the solution to the system?