Applications of Double Integrals

Section 15.5

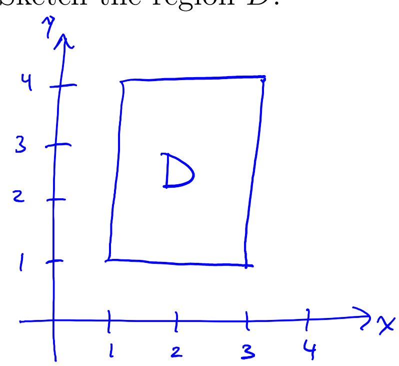
The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA \quad \text{and} \quad \bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA.$$

where the mass m is given by $m = \iint_D \rho(x,y)\,dA$.

1. Let $D = \{(x, y) \mid 1 \le x \le 3, 1 \le y \le 4\}$, and let $\rho(x, y) = 3y^2$. Use the following steps to find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

   (a) Sketch the region D.



   (b) Compute the mass of the lamina that occupes region D with density given by $\rho(x, y) = 3y^2$.

$$M = \iint_D \rho(x,y)\,dA = \int_1^3\int_1^4 3y^2\,dy\,dx = \int_1^3 63\,dx = 63x\Big|_1^3 = \boxed{126}$$

$$\int_1^4 3y^2\,dy = y^3\Big|_1^4 = 4^3 - 1^3 = 64 - 1 = 63$$

   (c) Compute the center of mass of the lamina using the above formulas.

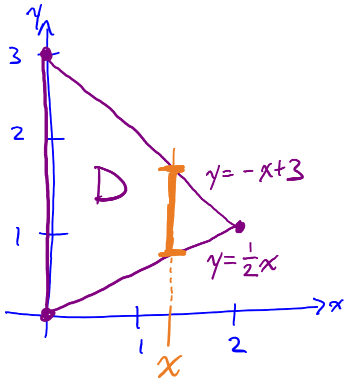
$$M_y = \int_1^3\int_1^4 x(3y^2)\,dy\,dx = 3\int_1^3 x\int_1^4 y^2\,dy\,dx = 3\int_1^4 y^2\,dy\int_1^3 x\,dx = 3\left[\tfrac{1}{3}y^3\right]_1^4\left[\tfrac{1}{2}x^2\right]_1^3$$

$$= 3\left(\frac{4^3}{3} - \frac{1^3}{3}\right)\left(\frac{3^2}{2} - \frac{1^2}{2}\right) = 3\cdot\frac{63}{3}\cdot\frac{8}{2} = 252$$

$$M_x = \int_1^3\int_1^4 y(3y^2)\,dy\,dx = \int_1^3 1\,dx\int_1^4 3y^3\,dy = 2\left[\tfrac{3}{4}y^4\right]_1^4 = \frac{3}{2}(4^4 - 1^4) = \frac{3(255)}{2}$$

center of mass: $(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{252}{126}, \frac{\frac{3(255)}{2}}{126}\right) = (2, 3.04)$

2. Let D be the triangular region with vertices $(0,0)$, $(2,1)$, $(0,3)$ and $\rho(x,y) = x + y$. Find the mass and center of mass of the region D by following a similar procedure as the previous problem.



Mass: $m = \iint_D \rho(x,y)\,dA = \int_0^2 \int_{\frac{1}{2}x}^{3-x} (x+y)\,dy\,dx = \int_0^2 \left(\frac{9}{2} - \frac{9}{8}x^2\right)dx = 6$

$$\int_{\frac{1}{2}x}^{3-x} (x+y)\,dy = \left[xy + \frac{1}{2}y^2\right]_{y=\frac{1}{2}x}^{y=3-x} = \left(x(3-x) + \frac{1}{2}(3-x)^2\right) - \left(x\left(\frac{1}{2}x\right) + \frac{1}{2}\left(\frac{1}{2}x\right)^2\right) = \frac{9}{2} - \frac{9}{8}x^2$$

Moments: $M_y = \int_0^2 \int_{\frac{1}{2}x}^{3-x} x(x+y)\,dy\,dx = \int_0^2 \left(\frac{9}{2}x - \frac{9}{8}x^3\right)dx = \frac{9}{2}$

$$M_x = \int_0^2 \int_{\frac{1}{2}x}^{3-x} y(x+y)\,dy\,dx = \int_0^2 \left(9 - \frac{9}{2}x\right)dx = 9$$

Center of mass: $(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{\frac{9}{2}}{6}, \frac{9}{6}\right) = \boxed{\left(\frac{3}{4}, \frac{3}{2}\right)}$

$$\int_0^2 \int_0^3 \int_0^4 2xy\,dz\,dy\,dx = \int_0^2 36x\,dx = 18x^2\Big|_{x=0}^{x=2} = 18(2^2 - 0^2) = \boxed{72}$$

$$\int_0^3 \int_0^4 2xy\,dz\,dy = \int_0^3 8xy\,dy = 4xy^2\Big|_{y=0}^{y=3} = 4x(3^2 - 0^2) = 36x$$

$$\int_0^4 2xy\,dz = 2xyz\Big|_{z=0}^{z=4} = 2xy(4-0) = 8xy$$

3. Suppose that the solid rectangular prism defined by $0 \le x \le 2$, $0 \le y \le 3$, and $0 \le z \le 4$ has density at a point (x,y,z) given by $\rho(x,y,z) = 2xy$. The mass of the solid can be computed by the *triple* integral

$$\int_0^2 \int_0^3 \int_0^4 2xy\;dzdydx.$$

Evaluate the integral above to find the mass of the rectangular prism..