Double Integrals over General Regions

Section 15.3

Let D be a region in the xy-plane of the form

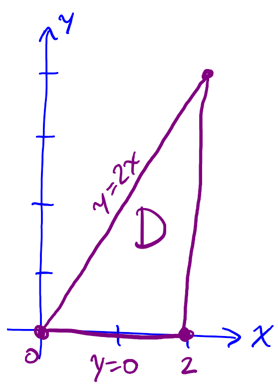
$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

If f is continuous on the region D, then

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

1. Let D be the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 4)$. Evaluate $\iint_D (x+y)\, dA$ by following the steps below.

   (a) Sketch the region D. What are $g_1(x)$ and $g_2(x)$? What are a and b?



$g_1(x) = 0$

$g_2(x) = 2x$

$a = 0$

$b = 2$

   (b) Use your answers from part (a) to write the double integral over D as an iterated integral.

$$\iint_D (x+y)\, dA = \int_0^2 \int_0^{2x} (x+y)\, dy\, dx$$

   (c) Evaluate the iterated integral you wrote in (b) by first evaluating the inside integral, and then evaluating the outside integral.

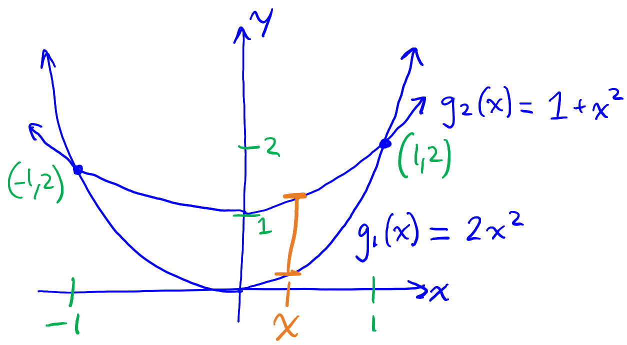
$$\int_0^{2x} (x+y)\, dy = \left[xy + \frac{1}{2}y^2\right]_{y=0}^{y=2x} = \left(x(2x) + \frac{1}{2}(2x)^2\right) - \left(x(0) + \frac{1}{2}(0)^2\right)$$

$$= 2x^2 + 2x^2 = 4x^2$$

$$\int_0^2 \int_0^{2x} (x+y)\, dy\, dx = \int_0^2 4x^2\, dx = \frac{4}{3}x^3 \Big|_0^2 = \frac{4}{3}(2^3 - 0^3) = \frac{4}{3}(8) = \boxed{\frac{32}{3}}$$

2. Evaluate $\iint_D (x+2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$, by following the steps below.

(a) Sketch the region D. What are $g_1(x)$ and $g_2(x)$?



(b) Determine the x values for when the two graphs intersect. These values represent a and b in the boxed formula on the previous page.

$$2x^2 = 1 + x^2$$
$$x^2 = 1$$
$$x = \pm 1$$

intersection points: $(1,2)$ and $(-1,2)$

$a = -1, \quad b = 1$

(c) Write the double integral over D as an iterated integral.

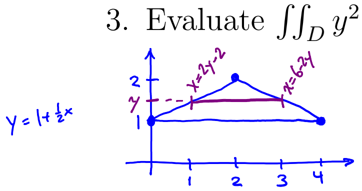
$$\iint_D (x+2y)\, dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

(d) Evaluate the iterated integral by integrating the inside integral first and then the outside integral last.

First:
$$\int_{2x^2}^{1+x^2} (x+2y)\,dy = \Big[xy + y^2 \Big]_{y=2x^2}^{y=1+x^2} = \left(x(1+x^2) + (1+x^2)^2\right) - \left(x(2x^2) + (2x^2)^2\right) =$$
$$= x + x^3 + 1 + 2x^2 + x^4 - 2x^3 - 4x^4 = -3x^4 - x^3 + 2x^2 + x + 1$$

Then:
$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx = \left[-\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x \right]_{x=-1}^{x=1}$$
$$= \left(-\frac{3}{5} - \cancel{\frac{1}{4}} + \frac{2}{3} + \cancel{\frac{1}{2}} + 1\right) - \left(\frac{3}{5} - \cancel{\frac{1}{4}} - \frac{2}{3} + \cancel{\frac{1}{2}} - 1\right) = -\frac{6}{5} + \frac{4}{3} + 2 = \frac{-18+20+30}{15} = \boxed{\frac{32}{15}}$$

3. Evaluate $\iint_D y^2\, dA$, where D is the triangle with vertices $(0,1)$, $(2,2)$ and $(4,1)$.



$$\iint_D y^2\, dA = \int_1^2 \int_{2y-2}^{6-2y} y^2\, dx\, dy = \int_1^2 (8y^2 - 4y^3)\, dy = \boxed{\frac{11}{3}}$$