Maximum and Minimum Values

Section 14.7

Theorem: If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exists there, then $f_x(a, b) =$ 0 and $f_y(a, b) =$ 0.

these are called CRITICAL POINTS

Second Derivatives Test: Suppose the second partial derivatives of f are continuous on a disk with center (a, b) and suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

(a,b) is a critical point

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local Maximum.
3. If $D < 0$, then $f(a, b)$ is a saddle point

If D=0, then the test offers no conclusion.

1. Find the critical points of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

   (a) Find f_x and f_y.

   $f_x(x,y) = 2x - 2$

   $f_y(x,y) = 2y - 6$

   (b) Solve the system of equations $f_x = 0 = f_y$. The corresponding solutions (x, y) are the critical points.

   $f_x(x,y) = 0$ when $2x = 2$, or $x = 1$

   $f_y(x,y) = 0$ when $2y = 6$, or $y = 3$

   CRITICAL POINT: $(1,3)$

   (c) Classify the critical points using the Second Derivatives Test.

   $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$

   $D = f_{xx} f_{yy} - (f_{xy})^2 = 4$

   At $(1,3)$: Since $D=4$ and $f_{xx}=2$

   (d) Notice (by completing the square) that $f(x, y) = (x - 1)^2 + (y - 3)^2 + 4$. How does this relate to your answer to the previous problem?

   The graph of $f(x,y)$ is a paraboloid that opens upward, so its only critical point must be a local minimum.

2. Find and classify[1] all the critical points for $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

(a) Find f_x and f_y.

$$f_x(x,y) = 6xy - 6x$$

$$f_y(x,y) = 3x^2 + 3y^2 - 6y$$

(b) Solve the system of equations $f_x = 0 = f_y$. The corresponding solutions (x, y) are the critical points.

$f_x = 0$ means $6xy - 6x = 0$, or $6x(y-1) = 0$, so $\underline{x=0}$ or $\underline{y=1}$

$f_y = 0$ means $3x^2 + 3y^2 - 6y = 0$

If $x=0$: $3y^2 - 6y = 3y(y-2) = 0$, so $y=0$ or $y=2$

If $y=1$: $3x^2 + 3(1)^2 - 6(1) = 3x^2 - 3 = 0$, so $x=-1$ or $x=1$.

CRITICAL POINTS: $(0,0)$, $(0,2)$, $(-1,1)$, $(1,1)$

(c) Classify the critical points using the Second Derivatives Test.

$f_{xx} = 6y-6$, $\quad f_x = 6y-6$, $\quad f_{xy} = f_{yx} = 6x$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = (6y-6)^2 - (6x)^2$$

At $(0,0)$: $D = 36$ and $f_{xx}(0,0) = -6$, so $(0,0)$ is a local max

At $(0,2)$: $D = 36$ and $f_{xx}(0,2) = 6$, so $(0,2)$ is a local min

At $(-1,1)$: $D = -36$, so $(-1,1)$ is a saddle point

At $(1,1)$: $D = -36$, so $(1,1)$ is a saddle point

[1] $(0, 0)$ - Local Max. $(0, 2)$ - Local Min. $(1, 1)$ - Saddle. $(-1, 1)$ - Saddle