Partial Derivatives

Sections 14.1 and 14.3

1. The wind-chill index W depends on temperature T and wind speed v:

		Wind Speed (km/hr)				
		5	10	15	20	25
Temperature	5	4	3	2	1	1
	0	-2	-3	-4	-5	-6
	-5	-7	-9	-11	-12	-12
	-10	-13	-15	-17	-18	-19
	-15	-19	-21	-23	-24	-25

Since W is a function of T and v, we can write $W = f(T, v)$.

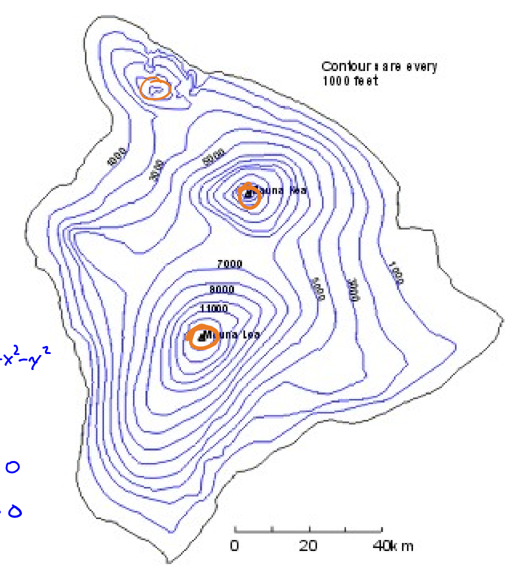
(a) What does $f(-15, 15)$ represent?

The wind chill at temperature $-15°C$ and wind speed 15 km/hr is $-23°C$.

(b) What is the meaning of the function $f(-10, v)$?

Gives the wind chill at a wind speed of v km/hr when the temperature is $-10° C$.

We often visualize a function of two variables by its **graph** or its **level curves**. If you know how to read a contour map, then you are familiar with the concept of level curves.



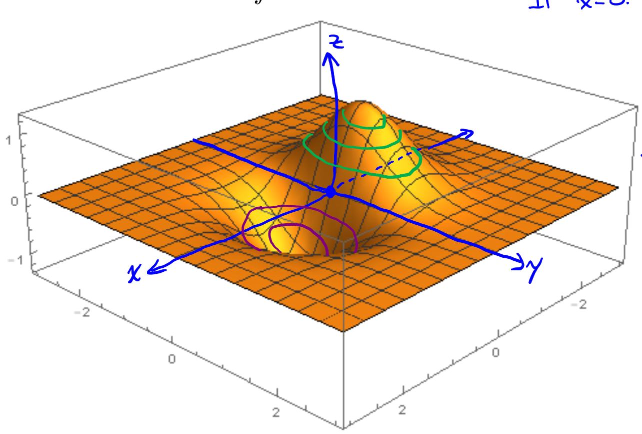
2. The graph of $f(x, y) = -3xe^{-x^2-y^2}$ is pictured below. Sketch the level curves of f.

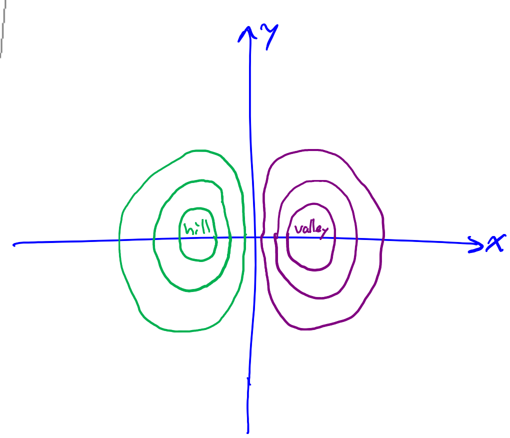
If $x=0$: $f(0,y) = -3(0)e^{-x^2-y^2}$

$f(0,y) = 0$

$f(1,0) = -3(1)e^{-1} < 0$

$f(-1,0) = -3(-1)e^{-1} > 0$





3. In the wind chill example, let $g(v) = f(-10, v)$. What does $g'(v)$ represent? Find an approximate value for $g'(15)$.

$g(v)$ gives the rate of change of the wind chill with respect to wind speed, for a temperature of $-10°C$

$$g'(15) \approx \frac{g(15)-g(10)}{15-10} = \frac{-17-(-15)}{5} = \frac{-2}{5} \ {}^{\circ}C/_{km/hr}$$

Let $f(x, y)$ be a function of two variables. Suppose we keep y fixed and consider the rate of change of f while x varies. This is the **partial derivative of f with respect to x**, is denoted $f_x(x, y)$, and is defined:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

Similarly, we can keep x fixed and consider the rate of change of f while y varies. This is the **partial derivative of f with respect to y**, is denoted $f_y(x, y)$, and is defined:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

4. Find the first partial derivatives of $f(x, y) = y^2\sqrt{x}$.

$$f_x(x,y) = y^2 \frac{1}{2\sqrt{x}} = \frac{y^2}{2\sqrt{x}} \qquad f_y(x,y) = 2y\sqrt{x}$$

5. Find *all* second partial derivatives of $f(x, y) = 2x^3y^2 + 4x^2y$.

$f_x = 6x^2y^2 + 8xy$ $\qquad f_{xx} = 12xy^2 + 8y$ $\qquad f_{yx} = 12x^2y + 8x$

$f_y = 4x^3y + 4x^2$ $\qquad f_{xy} = 12x^2y + 8x$ $\qquad$ same! $\qquad f_{yy} = 4x^3$

Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then:

$$f_{xy}(a,b) = f_{yx}(a,b)$$

6. For each of the following functions, find all second partial derivatives and verify that Clairaut's Theorem holds.

(a) $f(x, y) = 36 - x^2 - 4y^2$

$f_x(x,y) = -2x$ $\qquad f_{xx}(x,y) = -2$ $\qquad f_{xy}(x,y) = f_{yx}(x,y) = 0$

$f_y(x,y) = -8y$ $\qquad f_{yy}(x,y) = -8$

(b) $g(x, y) = e^{xy}$

$g_x(x,y) = ye^{xy}$ $\qquad g_{xx}(x,y) = y^2e^{xy}$ $\qquad g_{xy}(x,y) = g_{yx}(x,y) = e^{xy} + xye^{xy}$

$g_y(x,y) = xe^{xy}$ $\qquad g_{yy}(x,y) = x^2e^{xy}$

(c) $h(x, y) = \ln(x^2 + y^2)$

$h_x(x,y) = \frac{2x}{x^2+y^2}$ $\qquad h_{xx}(x,y) = \frac{(x^2+y^2)(2)-(2x)(2x)}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$ $\qquad h_{xy}(x,y) = h_{yx}(x,y) = \frac{-4xy}{(x^2+y^2)^2}$

$h_y(x,y) = \frac{2y}{x^2+y^2}$ $\qquad h_{yy}(x,y) = \frac{(x^2+y^2)(2)-(2y)(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$