Taylor Series

The **Taylor series** for a function $f(x)$ centered at $x = 0$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \cdots$$

1. Calculate $T_5(x)$, the Taylor polynomial of degree 5 that approximates $f(x) = \sin(x)$ near $x = 0$.

   (a) First calculate all derivatives of $f(x)$ up to order 5 and evaluate them at $x = 0$.

   $f(x) = \sin(x)$ $\quad f'''(x) = -\cos(x)$ $\quad f(0) = 0$ $\quad f'''(0) = -1$

   $f'(x) = \cos(x)$ $\quad f^{(4)}(x) = \sin(x)$ $\quad f'(0) = 1$ $\quad f^{(4)}(0) = 0$

   $f''(x) = -\sin(x)$ $\quad f^{(5)}(x) = \cos(x)$ $\quad f''(0) = 0$ $\quad f^{(5)}(0) = 1$

   (b) Now, use these values to calculate $\frac{f^{(n)}(0)}{n!}$ for $n = 0, 1, 2, 3, 4, 5$. These are called the Taylor coefficients (for the fifth order Taylor polynomial).

n	0	1	2	3	4	5
$\frac{f^{(n)}(0)}{n!}$	0	1	0	$\frac{-1}{3!} = \frac{-1}{6}$	0	$\frac{1}{5!} = \frac{1}{120}$

   (c) Use the numbers you calculated in the previous question to write out the fifth order Taylor polynomial (that is, the Taylor Series up to the x^5 term).

   $$T_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

   (d) Use the Taylor polynomial you just found to approximate the value of $\sin\left(\frac{\pi}{2}\right)$. What is the error? That is, how far is $T_5\left(\frac{\pi}{2}\right)$ from the true value of $\sin\left(\frac{\pi}{2}\right)$?

   $$\left.\begin{array}{l} T_5\left(\frac{\pi}{2}\right) = 1.00452 \\ \sin\left(\frac{\pi}{2}\right) = 1 \end{array}\right\} \text{error is } 0.00452$$

2. On another sheet of paper, calculate the fifth order Taylor Polynomial for $f(x) = \frac{1}{1+x}$

   for $f(x) = \frac{1}{1+x}$, $\quad T_5(x) = 1 - x + x^2 - x^3 + x^4 - x^5$

The Taylor series for a function $f(x)$ centered $x = a$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n \cdots$$

3. Using a similar procedure, calculate the the 5th order Taylor Polynomial for $f(x) = \sqrt{1+x}$ centered at $a = 1$.

$$f(1) = \sqrt{2},\ f'(1) = \frac{1}{2\sqrt{2}},\ f''(1) = \frac{-1}{8\sqrt{2}},\ f'''(1) = \frac{3}{32\sqrt{2}},\ f^{(4)}(1) = \frac{-15}{128\sqrt{2}},\ f^{(5)}(1) = \frac{105}{512\sqrt{2}}$$

$$T_5(x) = \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1) - \frac{\frac{1}{8\sqrt{2}}}{2}(x-1)^2 + \frac{\frac{3}{32\sqrt{2}}}{3!}(x-1)^3 - \frac{\frac{15}{128\sqrt{2}}}{4!}(x-1)^4 + \frac{\frac{105}{512\sqrt{2}}}{5!}(x-1)^5$$

$$= \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1) - \frac{1}{16\sqrt{2}}(x-1)^2 + \frac{1}{64\sqrt{2}}(x-1)^3 - \frac{5}{1024\sqrt{2}}(x-1)^4 + \frac{7}{4096\sqrt{2}}(x-1)^5$$

4. Suppose $g(x)$ is a function which has continuous derivatives and that $g(5) = 3$, $g'(5) = -2$, $g''(5) = 1$, $g'''(5) = -3$.

(a) What is the Taylor polynomial of degree 2 for g near 5? What is the Taylor polynomial of degree 3 for g near 5?

$$T_2(x) = 3 - 2(x-5) + \frac{1}{2}(x-5)^2$$

$$T_3(x) = 3 - 2(x-5) + \frac{1}{2}(x-5)^2 - \frac{3}{6}(x-5)^3$$

(b) Use the two Taylor polynomials that you just found to approximate $g(4.9)$.

$$T_2(4.9) = 3.205$$

$$T_3(4.9) = 3.2055$$

5. Find the third-degree Taylor polynomial for $f(x) = x^3 + 7x^2 - 5x + 1$ about $x = 0$. What do you notice?

$$T_3(x) = 1 - 5x + 7x^2 + x^3 = f(x)$$