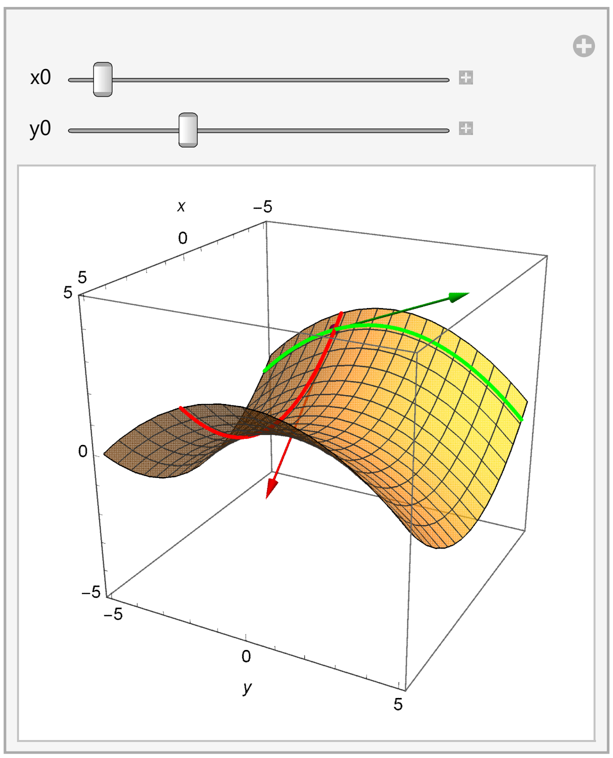
Illustration of Partial Derivatives

In[1]:=
```
f[x_, y_] := (x^2 - y^2) / 10;
Manipulate[
 Show[{
   Plot3D[f[x, y], {x, -5, 5}, {y, -5, 5}, BoxRatios → Automatic,
    PlotRange → {-5, 5}, PlotStyle → Opacity[0.7], AxesLabel → Automatic],
   ParametricPlot3D[{{t, y0, f[t, y0]}, {x0, t, f[x0, t]}},
    {t, -5, 5}, PlotStyle → {Red, Green}],
   Graphics3D[{Purple, Sphere[{x0, y0, f[x0, y0]}, 0.2] , Red,
     Arrow[Tube[{{x0, y0, f[x0, y0]}, {x0 + 5, y0, f[x0, y0] + x0}}, 0.04]], Green,
     Arrow[Tube[{{x0, y0, f[x0, y0]}, {x0, y0 + 5, f[x0, y0] - y0}}, 0.04]]}]
  }]
 , {x0, -5, 5}, {y0, -5, 5}]
```

Out[2]=



Tangent Planes

Calc I reminder: Remember that the equation for a line (in 2 dimensions) is

$$y = y_0 + m(x - x_0).$$

point: (x_0, y_0)
slope: m

We use this to find the equation of the line tangent to the graph of $y = f(x)$ and $x = x_0$. The slope of this line is $m = f'(x_0)$, so we get

slope

$$y = y_0 + f'(x_0)(x - x_0).$$

$y_0 = f(x_0)$

1. Find the equation of the line tangent to $f(x) = x^2$ at $x = 2$. $y_0 = f(2) = 4$

$f'(x) = 2x$ at $x = 2$: $f'(2) = 4$

point-slope form: $y = 4 + 4(x - 2)$

slope-intercept form: $y = 4x - 4$

2. Now imagine a 3D surface (such as a sphere). At any point on the sphere, there are *many* different lines that are tangent. So it doesn't make sense to talke about "THE tangent line." What kind of "tangent object" should we really be looking for?

We want a tangent plane to a surface at a point.

3. Group Conjecture: It is often quite easy to find a 3-dimensional version of something if you know the 2-dimensional version. Fill in the boxes with what you think should be there.

2-dimensional version	$\Rightarrow$	3-dimensional version
2D distance formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	$\Rightarrow$	**3D distance formula** $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
equation of a line $ax + by = d$	$\Rightarrow$	**equation of a plane** $ax + by + cz = d$
equation for a circle $x^2 + y^2 = r^2$	$\Rightarrow$	**equation for a sphere** $x^2 + y^2 + z^2 = r^2$
line tangent to $f(x)$ **at** $x = x_0$ $y = y_0 + f'(x_0)(x - x_0)$	$\Rightarrow$	**plane tangent** $f(x, y)$ **at** $(x, y) = (x_0, y_0)$ $z = z_0 + \boxed{f_x(x_0, y_0)}\left(x - \boxed{x_0}\right) + \boxed{f_y(x_0, y_0)}\left(y - \boxed{y_0}\right)$

4. Find the equation of the plane that is tangent to $f(x,y) = 2x^2 + y^2 - 3y$ at $(x,y) = (1,1)$.

$z_0 = f(1,1) = 2(1)^2 + (1)^2 - 3(1) = 0$

$f_x = 4x$, so $f_x(1,1) = 4$

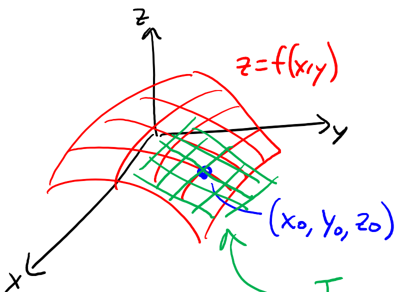
$f_y = 2y - 3$, so $f_y(1,1) = -1$

equation of tangent plane:

$z = 0 + 4(x-1) - 1(y-1)$

$z = \underbrace{4x - y - 3}$

5. Use the tangent plane you found in #4 to estimate the value of $f(1.1, 1.1)$ without actually plugging these values into $f(x,y)$ itself.



$f(1.1, 1.1) \approx 4(1.1) - (1.1) - 3 = 0.3$

close!

Actual value:

$f(1.1, 1.1) = 2(1.1)^2 + (1.1)^2 - 3(1.1) = 0.33$

Tangent plane approximates $f(x,y)$ near (x_0, y_0)

6. Find the equation of the plane that is tangent to $f(x,y) = x\sin(x+y)$ at $(x,y) = (\pi,\pi)$.

$f(\pi,\pi) = \pi \sin(2\pi) = 0$

$f_x(x,y) = \sin(x+y) + x\cos(x+y)$, $\quad f_x(\pi,\pi) = \pi$

$f_y(x,y) = x\cos(x+y)$, $\quad f_y(\pi,\pi) = \pi$

plane: $z = 0 + \pi(x-\pi) + \pi(y-\pi)$ or $z = \pi x + \pi y - 2\pi^2$

7. Find the equation of the plane tangent to $f(x,y) = 3 - 2x + 5y$. Simplify as much as possible.

Does the result surprise you?

At $(2,3)$: $f(2,3) = 3 - 2(2) + 5(3) = 14$

Same!

$f_x = -2$ so $f_x(2,3) = -2$

$f_y = 5$ so $f_y(2,3) = 5$

plane: $z = 14 - 2(x-2) + 5(y-3)$

$z = 14 - 2x + 4 + 5y - 15$

$\boxed{z = 3 - 2x + 5y}$

Notes: The tangent plane is also called the **linearization** of $f(x,y)$ at a point (x_0, y_0) and written

$$L(x,y) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}.$$

equation of the tangent plane

The tangent plane only exists at points (x_0,y_0) where the partial derivatives f_x and f_y exist and are continuous.

8. Find the linearization of $f(x,y) = \sqrt{10 - x^2 - 3y^2}$ at $(2,1)$ and use it to approximate $f(1.9, 1.1)$.

$f(2,1) = \sqrt{10-(2)^2-3(1)^2}$

$f_x(x,y) = \frac{1}{2}(10-x^2-3y^2)^{-1/2}(-2x) = -x(10-x^2-3y^2)^{-1/2}$, $f_x(2,1) = -2(3)^{-1/2}$

$f_y(x,y) = \frac{1}{2}(10-x^2-3y^2)^{-1/2}(-6y) = -3y(10-x^2-3y^2)^{-1/2}$, $f_y(2,1) = -3(3)^{-1/2}$

Linearization: $L(x,y) = \sqrt{3} - \frac{2}{\sqrt{3}}(x-2) - \frac{3}{\sqrt{3}}(y-1)$ and $L(1.9,1.1) = 1.674$

9. Find the linearization of $z = y\ln(x)$ at $(1,4,0)$.

Partial derivatives: $z_x = \frac{y}{x}$ and $z_y = \ln(x)$

At $x=1$, $y=4$: $z_x(1,4) = 4$ $z_y(1,4) = 0$

Linearization:

$L(x,y) = 0 + 4(x-1) + 0(y-4)$

$L(x,y) = 4x - 4$

10. Suppose that your friend claims that the equation of the tangent plane to the graph of $f(x,y) = x^3 - y^2$ at the point $(4,5)$ is

$$z = 39 + 3x^2(x-4) - 2y(y-5).$$

(a) Why is this not possibly the equation of a tangent plane?

Think about this for next time!

(b) What mistake did your friend make?

(c) What is the correct equation of the tangent plane?