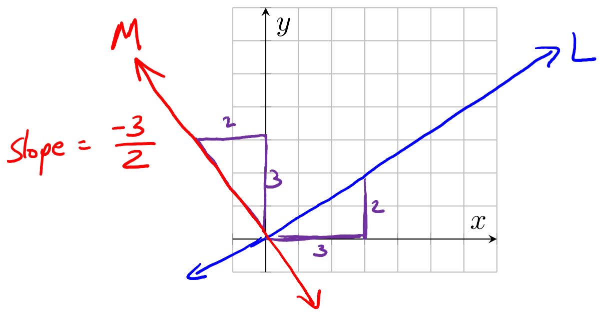
Dot Product

Recall: A vector encodes the *change* in x, y, and z from one point to another.
For example, the vector from point $A = (1, 2, 3)$ to point $B = (8, 10, 2)$ is:

$$\overrightarrow{AB} = \langle\ 7\ ,\ 8\ ,\ -1\ \rangle$$

1. (a) In 2D, draw the line L through the points $A = (0, 0)$ and $B = (3, 2)$. What is the slope of L?



Slope: $\frac{2}{3}$

(b) What is the slope of the line M that is *perpendicular* to L?

$\frac{-3}{2}$

☞ Feel free to draw M, too! You might want to draw M so it goes through the origin.

(c) Find a vector (in 2D) that is *parallel* to L. How is your vector related to the slope of L?

$\langle 3, 2\rangle$ or $\langle -3, -2\rangle$ or $\langle 6, 4\rangle$ or $\langle 300, 200\rangle$ or ...

☞ Maybe use A and B?

(d) Now, try to find a vector that is *parallel* to the line M (and thus perpendicular to L).

$\langle 2, -3\rangle$ or $\langle -2, 3\rangle$ or $\langle -200, 300\rangle$ or ...

Definition Alert: The **dot product** is a calculation done with two vectors. The result is a number.

vectors   number (scalar)

- In 2D: $\langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = a_1b_1 + a_2b_2$
- In 3D: $\langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3$

2. (a) What is $\langle 3, 1\rangle \cdot \langle 1, 2\rangle$?

$\langle 3,1\rangle \cdot \langle 1,2\rangle = 3\cdot 1 + 1\cdot 2 = 5$

(b) What is $\langle -2, 1, 0\rangle \cdot \langle 1, 0, 1\rangle$?

$\langle -2,1,0\rangle \cdot \langle 1,0,1\rangle = -2\cdot 1 + 1\cdot 0 + 0\cdot 1 = -2$

(c) What is $\langle 3, 2\rangle \cdot \langle 2, -3\rangle$? ← vectors from part 1, perpendicular

$\langle 3,2\rangle \cdot \langle 2,-3\rangle = 3\cdot 2 + 2(-3) = 6 - 6 = 0$

(d) What is $\langle a, b, c\rangle \cdot \langle a, b, c\rangle$? How is this related to $|\langle a, b, c\rangle|$? ← length, $= \sqrt{a^2+b^2+c^2}$

☞ Remember, $|\langle a, b, c\rangle|$ is the length of $\langle a, b, c\rangle$.

$\langle a,b,c\rangle \cdot \langle a,b,c\rangle = a^2 + b^2 + c^2 = |\langle a,b,c\rangle|^2$

3. Make a conjecture with your table partners:

Vectors **u** and **v** are perpendicular (or *orthogonal*) exactly when $\mathbf{u}\cdot\mathbf{v}$ equals $\underline{\quad 0 \quad}$.

4. Research: If **u** and **v** are parallel vectors, how does $\mathbf{u}\cdot\mathbf{v}$ relate to $|\mathbf{u}|$ and $|\mathbf{v}|$?

Experiment with some parallel vectors!

$\vec{u} = \langle 3,2\rangle \qquad |\vec{u}| = \sqrt{3^2+2^2} = \sqrt{13}$

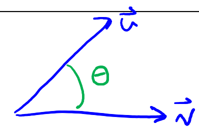
$\vec{v} = \langle 9,6\rangle \qquad |\vec{v}| = \sqrt{9^2+6^2} = \sqrt{117} = \sqrt{9(13)} = 3\sqrt{13}$

$\vec{u}\cdot\vec{v} = 9\cdot 3 + 2\cdot 6 = 27 + 12 = 39 \qquad 39 = (\sqrt{13})(3\sqrt{13})$

If $\vec{u}, \vec{v}$ are parallel: $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$

If θ is the angle between vectors **v** and **u**, then:

$$\mathbf{v}\cdot\mathbf{u} = |\vec{v}|\,|\vec{u}|\,\cos\theta$$



5. Let $\mathbf{u} = \langle 2,0,4\rangle$ and $\mathbf{v} = \langle -1,2,3\rangle$. If θ is the angle between **u** and **v**, find $\cos\theta$.

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$\langle 2,0,4\rangle\cdot\langle -1,2,3\rangle = |\langle 2,0,4\rangle||\langle -1,2,3\rangle|\cos\theta$

$-2+0+12 = \sqrt{4+0+16}\,\sqrt{1+4+9}\,\cos\theta$

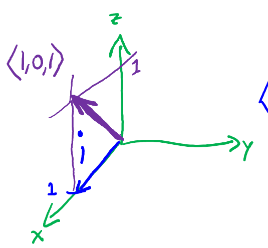
$10 = \sqrt{20}\,\sqrt{14}\,\cos\theta$

so $\cos\theta = \dfrac{10}{\sqrt{20}\,\sqrt{14}}$

$\theta = \arccos\left(\dfrac{10}{\sqrt{20}\sqrt{14}}\right)$

6. Find the angle between the vectors $\langle 1,0,1\rangle$ and **i**.

$\mathbf{i} = \langle 1,0,0\rangle$



$\langle 1,0,1\rangle\cdot\langle 1,0,0\rangle = |\langle 1,0,1\rangle||\langle 1,0,0\rangle|\cos\theta$

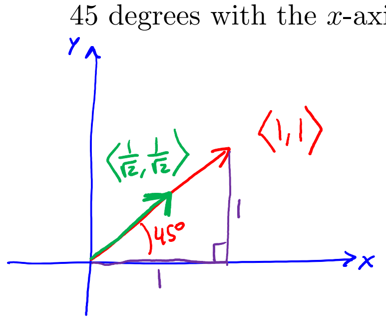
$1 = \sqrt{2}\,\sqrt{1}\,\cos\theta$

so $\cos\theta = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\theta = \dfrac{\pi}{4}$ or $45°$

7. Find a 2D vector that makes an angle of 60 degrees with the x-axis.

8. A vector of length 1 is called a **unit vector**. Find a 2D unit vector that makes an angle of 45 degrees with the x-axis.



$\langle 1,1\rangle$ has the right angle, but is not a unit vector

$\langle 1,1\rangle$ has length $\sqrt{1^2+1^2} = \sqrt{2}$

Divide the vector by its length:

$\left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right\rangle$

To find a unit vector in the same direction as a given vector $\vec{v}$, divide $\vec{v}$ by its length.

Given two vectors $\mathbf{u}$ and $\mathbf{v}$, we might want to find the **component** of $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$\text{comp}_{\mathbf{v}}\mathbf{u} =$$

The **projection** of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\text{proj}_{\mathbf{v}}\mathbf{u} =$$

9. Draw a picture to illustrate $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{comp}_{\mathbf{v}}\mathbf{u}$.

10. Find the scalar and vector projections of $\mathbf{u} = \langle 2, 3\rangle$ onto $\mathbf{v} = \langle 1, 4\rangle$.

11. Find the scalar and vector projections of $\mathbf{u} = 2\mathbf{j} + \mathbf{k}$ onto $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

12. Show that the vector $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{v}$.