# Using Taylor Series

Here are common functions and their Maclaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{for } -1 < x < 1$$

← Geometric series $a=1$, $r=x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{for all } x$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all } x$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x$$

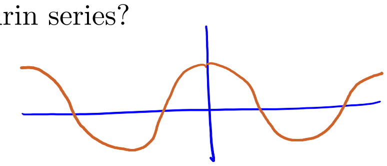
$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } -1 < x \le 1$$

**1.** What type of symmetry does $\cos(x)$ have? How does this appear in its Maclaurin series?

y-axis symmetry! "even symmetry"

$\cos(-x) = \cos(x)$

All powers of $x$ in the series are even!

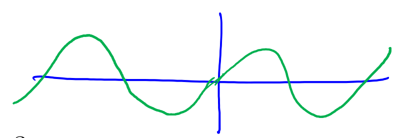


**2.** What type of symmetry does $\sin(x)$ have? How does this appear in its Maclaurin series?

rotational symmetry about the origin — "odd symmetry"

$\sin(-x) = -\sin(x)$

Powers of $x$ are odd in the series!



**3.** How are the Maclaurin series for $e^x$, $\sin(x)$, and $\cos(x)$ related to each other?

The terms look the same!

Except that $e^x$ has all + terms.

$\boxed{e^{ix} = \cos(x) + i\sin(x) \quad \text{where } i = \sqrt{-1}}$ EULER'S FORMULA

**4.** All of the series below converge. Find the sum of each.

Do these look at ALL like any of the Maclaurin series we have studied?

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} = \underset{n=0}{1} + \underset{n=1}{\frac{2}{1!}} + \underset{n=2}{\frac{2^2}{2!}} + \underset{n=3}{\frac{2^3}{3!}} + \cdots = e^2$

the series for $e^x$, when $x=2$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{36^n (2n)!} = \underset{n=0}{1} - \underset{n=1}{\frac{\pi^2}{36(2!)}} + \underset{n=2}{\frac{\pi^4}{36^2(4!)}} - \underset{n=3}{\frac{\pi^6}{36^3(6!)}} + \cdots$

Hint: $36 = 6^2$

$$= 1 - \frac{\left(\frac{\pi}{6}\right)^2}{2!} + \frac{\left(\frac{\pi}{6}\right)^4}{4!} - \frac{\left(\frac{\pi}{6}\right)^6}{6!} + \cdots = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

(c) $1 + (\ln 3) + \dfrac{(\ln 3)^2}{2!} + \dfrac{(\ln 3)^3}{3!} + \dfrac{(\ln 3)^4}{4!} + \dfrac{(\ln 3)^5}{5!} + \cdots = e^{\ln(3)} = 3$

(d) $e - \frac{e^2}{2} + \frac{e^3}{3} - \frac{e^4}{4} + \frac{e^5}{5} - \frac{e^6}{6} + \cdots$

**5.** (a) Write down the 5th degree Maclaurin polynomial for $f(x) = \sin x$. Then, plug $x = 1$ into the polynomial and see what you get. You can use a computer.

$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ $\qquad \sin(1) = 0.8415$

$\sin(1) \approx 1 - \frac{1}{6} + \frac{1}{120} = \frac{120 - 20 + 1}{120} = \frac{101}{120} = 0.8416$

(b) Write down the 5th degree Taylor polynomial for $f(x) = \sin x$ centered at $x = \frac{\pi}{2}$. Then, plug $x = 1$ into the polynomial and see what you get. You can use a computer.

(c) **Discuss with your table partners:** Without actually calculating $\sin(1)$, which of (a) or (b) do you think is closer to the actual value of $\sin(1)$? Why?

☞ After you pick, check if your intuition was correct or not. Use a computer, phone, or calculator.

**6.** (a) Find the Maclaurin series for $f(x) = \arctan(x)$.

☞ Hint: $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

(b) Evaluate your series at $x = 1$. How does this give you an approximation of $\pi$?