Power Series

1. Can you tell if $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ converges or diverges? How do you know?

☞ If it converges, can you figure out what it converges to?

geometric series: $a = 1$, $r = \frac{2}{3}$

converges to $\frac{a}{1-r} = \frac{1}{1-\frac{2}{3}} = 3$

2. Can you tell if $\sum_{n=0}^{\infty} \left(\frac{-4}{3}\right)^n$ converges or diverges? How do you know?

☞ If it converges, can you figure out what it converges to?

geometric series: $a = 1$, $r = \frac{-4}{3}$

since $r < -1$, the series diverges

3. Research Question: What values of x can be substituted into $\sum_{n=0}^{\infty} x^n$ in order for the series to converge? For the values of x you find, what is the actual sum of the series?

☞ Hopefully there is an x in your answer, since it depends on what x is.

Converges for:
$-1 < x < 1$, since this is a geometric series with common ratio x

If $-1 < x < 1$, then the series converges to $\frac{1}{1-x}$.

$\boxed{\text{Note: } 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.}$

4. Spicy Research Question: What values of x can be substituted into $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ in order for the series to converge? For the values of x you find, what is the actual sum of the series?

Also a geometric series. For convergence, we need $-1 < \frac{x}{2} < 1$, or $-2 < x < 2$.

If $-2 < x < 2$, then the series converges to $\frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$.

5. Our goal is to study the *ratio* between two consecutive terms of a series! Suppose our series is $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$. Let $a_n = \left(\frac{2}{3}\right)^n$.

$a_0 = 1$, $a_1 = \frac{2}{3}$, $a_2 = \left(\frac{2}{3}\right)^2$, ...

(a) Calculate the following ratios: $\frac{a_1}{a_0}, \frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \frac{a_5}{a_4}, \frac{a_6}{a_5}, \frac{a_7}{a_6}$. Do you notice any patterns?

$\frac{a_1}{a_0} = \frac{\frac{2}{3}}{1} = \frac{2}{3}$, $\frac{a_2}{a_1} = \frac{\left(\frac{2}{3}\right)^2}{\left(\frac{2}{3}\right)} = \frac{2}{3}$, $\frac{a_3}{a_2} = \frac{\left(\frac{2}{3}\right)^3}{\left(\frac{2}{3}\right)^2} = \frac{2}{3}$, ...

(b) Pick your favorite value of n that is really large. Then, calculate $\frac{a_{n+1}}{a_n}$.

☞ Have each person at your table pick a different large number.

$n = 648$

$$\frac{a_{649}}{a_{648}} = \frac{\left(\frac{2}{3}\right)^{649}}{\left(\frac{2}{3}\right)^{648}} = \frac{2}{3}$$

(c) What do you think happens to $\frac{a_{n+1}}{a_n}$ as n gets larger and larger?

$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)^n}$ is always $\frac{2}{3}$

$$\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n = 0 + \frac{2}{3} + 2\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + 4\left(\frac{2}{3}\right)^4 + \cdots$$

6. Now suppose our series is $\displaystyle\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n$. Let $a_n = n\left(\frac{2}{3}\right)^n$.

$a_{n+1} = (n+1)\left(\frac{2}{3}\right)^{n+1}$

(a) Calculate the following ratios:

$$\frac{a_1}{a_0}, \frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \frac{a_5}{a_4}, \frac{a_6}{a_5}, \frac{a_7}{a_6}, \frac{a_8}{a_7}$$

Do you notice any patterns?

(b) Pick your favorite value of n that is really large. Then, calculate $\dfrac{a_{n+1}}{a_n}$.

☞ Have each person at your table pick a different large number.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\left(\frac{2}{3}\right)^{\cancel{n+1}\ 1}}{n\cancel{\left(\frac{2}{3}\right)^n}} = \frac{(n+1)\left(\frac{2}{3}\right)}{n} = {+1}\left(\frac{2}{3}\right)$$

(c) What do you think happens to $\dfrac{a_{n+1}}{a_n}$ as n gets larger and larger?

$\dfrac{a_{n+1}}{a_n}$ gets closer to $\dfrac{2}{3}$

(d) Try verifying your answer to part (c) by computing $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

☞ Substitute in the formulas for a_n and a_{n+1}.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n}\left(\frac{2}{3}\right) = \frac{2}{3}$$

7. I just heard somebody say

"The series $\displaystyle\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n$ behaves *a lot like* a geometric series."

Discuss with the people at your table:

(a) What does this mean?

For big n, $\frac{a_{n+1}}{a_n}$ approaches $\frac{2}{3}$, so there is almost a common ratio between the terms for big n.

(b) Do you think that $\displaystyle\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n$ converges or diverges?

We think this series converges, since it's almost the same as the geometric series $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ for big n.

(c) For what values of x do you think $\displaystyle\sum_{n=0}^{\infty} nx^n$ converges?.

☞ You already did the problem for $x = \frac{2}{3}$

If $-1<x<1$, then this series is almost the same as a convergent geometric series (for large n).

The Ratio Test: Suppose you want to check if a series $\sum_{n=0}^{\infty} a_n$ converges or diverges.

- If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$ or does not exist, then the test is inconclusive.

8. Now suppose our series is $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{x^n}{2n-1}$.

We will come back to this on Friday!

(a) Write out the first five terms of the series.

☞ The variable x should appear in your terms.

(b) Find a formula for a_{n+1}.

(c) What is $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$?

(d) For what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2n-1}$ converge?

9. Now suppose our series is $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

☞ Cool fact: $0! = 1$

(a) Write out the first five terms of the series.

☞ The variable x should appear in your terms.

(b) Find a formula for a_{n+1}.

(c) What is $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$?

(d) For what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?