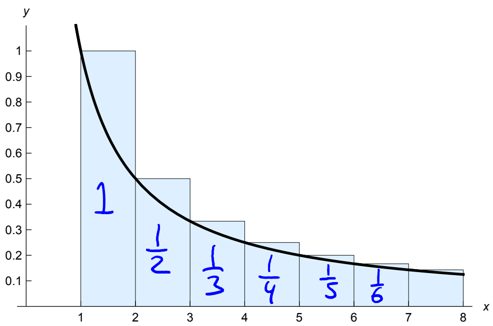
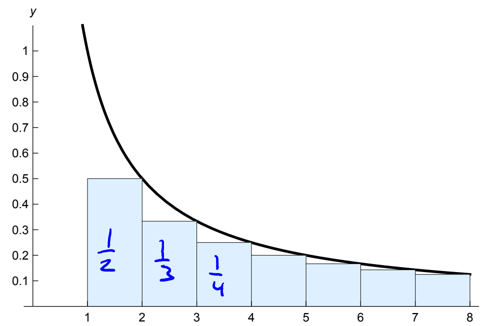
Convergence of Series

1. The following two diagrams show rectangles that approximate $\int_1^\infty \frac{1}{x}\,dx$.





(a) What series results when you add the areas of the left rectangles?

HARMONIC SERIES $\rightarrow$ $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$

DIVERGES!

(b) What series results when you add the areas of the right rectangles?

☜ Careful, it's *slightly* different

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n}$

(c) How can your knowledge about $\int_1^\infty \frac{1}{x}\,dx$ allow you to conclude whether $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges?

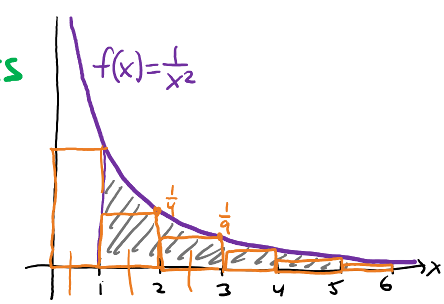
$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}dx = \lim_{b\to\infty} \ln x \Big|_1^b = \lim_{b\to\infty} (\ln b - 0) =$ diverges (∞)

Area of the rectangles is bigger than the area under $\frac{1}{x}$.
Since $\int_1^\infty \frac{1}{x}dx$ diverges, $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

2. Now consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$. What integral could you consider to figure out whether this series converges or diverges?

series $\rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ CONVERGES to $\frac{\pi^2}{6}$

series is smaller than the integral $\rightarrow$ $\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty} \int_1^b x^{-2}dx = \lim_{b\to\infty} -x^{-1}\Big|_1^b = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1$



$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$

The Integral Test: Suppose you have a *positive, continuous, and decreasing* function $f(x)$, and $f(n) = a_n$ for all n. Then,

The series $\sum_{n=1}^{\infty} a_n$ converges EXACTLY WHEN the integral $\int_1^\infty f(x)$ converges.

3. Let's look at the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n^2+4)^2} = \frac{2}{25} + \frac{4}{64} + \frac{6}{169} + \frac{8}{400} + \cdots$.

(a) Is the function $f(x) = \frac{2x}{(x^2+4)^2}$ continuous for x values in $[1, \infty)$?

Yes, the denominator is never zero.

(b) Is the function $f(x) = \frac{2x}{(x^2+4)^2}$ positive for x values in $[1, \infty)$?

Yes.

(c) Is the function $f(x) = \frac{2x}{(x^2+4)^2}$ decreasing (eventually) for x values in $[1, \infty)$? Yes.

(d) IF the answers to (a),(b),(c) are YES, you may TRY using the integral test. Does

$$\int_2^{\infty} \frac{2x}{(x^2+4)^2}\, dx \qquad \text{converge or diverge?}$$

$u = x^2+4$
$du = 2x\,dx$

$$\int_8^{\infty} \frac{du}{u^2} = \lim_{b\to\infty} \frac{-1}{u}\Big|_8^b = \lim_{b\to\infty}\left(\frac{-1}{b} + \frac{1}{8}\right) = \frac{1}{8}$$

Integral converges, and so does the series.

4. (a) If $n \geq 1$, then $\dfrac{1}{n^2+n}$ is bigger/smaller (circle one) than $\dfrac{1}{n^2}$.

(b) The sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is bigger/smaller (circle one) than the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(c) Why can you now conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges?

You don't need to know what a series converges TO in order to know that it converges.

5. Can you use the method from problem 4 to find out whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges or diverges?

6. Can you use the method from problem 4 to show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$ diverges?