Introduction to Series

1. We know that the number 0.3 can also be written as $\frac{3}{10}$.
We also know that the number 0.33 can also be written as $\frac{3}{10} + \frac{3}{100}$.

Written as a single fraction, $\frac{33}{100}$.

(a) Try to write the numbers 0.333 and 0.3333 in a similar fashion. Then, write each one as a single fraction.

$$0.333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$$

SEQUENCE: a list of terms

$$0.3333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000}$$

(b) Now try to write the number $y = 0.333\overline{3}$ in a similar fashion. What happens? Can you write y as a single fraction?

The "bar" over the last 3 means it just keeps going ... FOREVER ... and ever and ever and ever.

SERIES: a sum of terms

$$y = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} + \frac{3}{1000000} + \cdots = \frac{1}{3}$$

$\frac{3}{10^5}$ $\frac{3}{10^6}$

$$y = \frac{1}{3}$$

(c) In the previous part, you should have gotten a sum of *infinitely many* numbers. Can you come up with a formula that describes the n-th number in the sum?

If $a_n = a_{n-1} + \frac{3}{10^n}$, with $a_1 = \frac{3}{10}$, then a_n is the sum of the first n terms of the series.

Sum of n terms: $\sum_{i=1}^{n} \frac{3}{10^i} = \frac{3}{10} + \frac{3}{100} + \cdots + \frac{3}{10^n}$ $\qquad \frac{3}{10^n}$

2. In the previous problem, I really hope you found out that

GEOMETRIC SERIES:

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} + \frac{3}{1000000} + \frac{3}{10000000} + \cdots = \frac{1}{3}$$

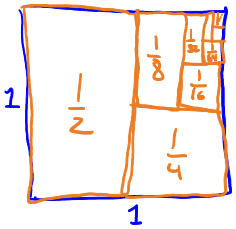
(a) Let's play. (Some call it research!) What do you think the answer is for this one?

Can you come up with a formula that describes the n-th number in the sum?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \cdots$$

Keep track of HOW you decide on your answer.

Sequence of partial sums: $\frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots, \frac{255}{256}, \ldots \to 1$

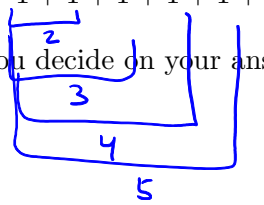


(b) Let's play more! What do you think the answer is for this one?

Can you come up with a formula that describes the n-th number in the sum?

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \cdots$$

Keep track of HOW you decide on your answer.



partial sums: 1, 2, 3, 4, 5, ... diverges

(c) Let's play EVEN more! What do you think the answer is for this one?

Can you come up with a formula that describes the n-th number in the sum?

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

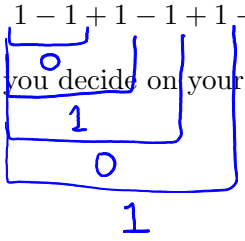
Keep track of HOW you decide on your answer.

0

1

0

1



partial sums: 1, 0, 1, 0, 1, 0, ... diverges

(d) *Spicy!* What do you think the answer is for this one?

Can you come up with a formula that describes the n-th number in the sum?

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

? We will come back to this.

3. Can you find an infinite series that converges to $\frac{5}{11}$?

Major Hint: What is the decimal representation of $\frac{5}{11}$?

$$\frac{5}{11} = 0.4545\overline{45}$$

$$0.4 + 0.05 + 0.004 + 0.0005 + 0.00004 + 0.000005 + \cdots = \frac{5}{11}$$

$$0.45 + 0.0045 + 0.000045 + \cdots = \frac{5}{11}$$

$\times \frac{1}{100}$ $\times \frac{1}{100}$

4. Can you find a formula for the terms of the following series?

$$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \frac{2}{243} + \cdots$$

What does the series converge to?

Geometric Series

Convergence of Series: Consider the infinite series

$$a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n.$$

The **sequence of partial sums** is the sequence $s_1, s_2, s_3, s_4, \ldots$, where

$$s_n = \text{ the sum of the first } n \text{ terms of the series } = a_1 + a_2 + \cdots + a_n.$$

If $\lim_{n\to\infty} s_n$ equals some number S, then we say that the series **converges** and S is its sum. Otherwise, we say that the series **diverges**.

1. Here is a cool series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \cdots$$

(a) Experiment by adding up several terms of the series. Can you find a formula for s_n?

$s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$, $s_4 = \frac{15}{16}, \ldots$ $s_n = \frac{2^n - 1}{2^n}$

(b) What happens to s_n as n gets HUGE?

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$

(c) Does $\sum_{n=1}^{\infty}$ converge or diverge? If it converges, what is its sum?

$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ converges

Definition: A series is called a **geometric series** if each term is always r times the previous term. A geometric series can be written like this:

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + \cdots = \sum_{n=0}^{\infty} ar^n.$$

Notice that each term is exactly r times the previous term. (r is called the *common ratio*)

2. Identify a and r for each of the following geometric series:

(a) $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \frac{2}{243} + \cdots$ $a = 2$, $r = \frac{1}{3}$

Use $n = 0$ as your starting value of n.

n^{th} term: $a_n = 2\left(\frac{1}{3}\right)^n$, with n starting at zero

(b) $5 - \frac{5}{3} + \frac{5}{9} - \frac{5}{27} + \frac{5}{81} - \frac{5}{243} + \cdots$ $a = 5$, $r = -\frac{1}{3}$, $a_n = 5\left(-\frac{1}{3}\right)^n$

(c) $\frac{1}{3} + \frac{2}{3} + \frac{4}{3} + \frac{8}{3} + \frac{16}{3} + \frac{32}{3} + \cdots$ $a = \frac{1}{3}$, $r = 2$, $a_n = \frac{1}{3}(2)^n$

3. Now we will find out which geometric series converge. To do this, look at the sequence of partial sums:

$$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^n$$

(a) What is $r \cdot s_n$?

$$r \cdot s_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n+1}$$

(b) You might notice that s_n and $r \cdot s_n$ are VERY similar. What happens when you subtract them?

$$s_n - r \cdot s_n = (a + ar + ar^2 + ar^3 + \cdots + ar^n) - (ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n+1})$$

(c) Use the above to solve for s_n.

$$s_n - r s_n = a - ar^{n+1}$$

$$s_n(1-r) = a(1-r^{n+1})$$

Formula for the partial sum of a geometric series →

$$s_n = \frac{a(1-r^{n+1})}{1-r}$$

Look, you found a formula for s_n. WOW!

(d) In order for the geometric series to converge, you need $\lim_n \to \infty s_n$ to be a number (not infinity or oscillating). Try a few specific values for r and see if the limit exists!

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r} = \begin{cases} \frac{a}{1-r} & \text{if } -1 < r < 1 \\ \text{diverges} & \text{otherwise} \end{cases}$$

Try positive, negative, fractions, etc.

(e) For what values of r does the geometric series converge? What does it converge TO?

4. Determine if the series converges or diverges. If the series converges, find what it converges to.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n = -\frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$

$a = -\frac{2}{3}, \quad r = -\frac{2}{3}$

Sum: $\frac{a}{1-r} = \frac{-\frac{2}{3}}{1-(-\frac{2}{3})} = \frac{-\frac{2}{3}}{\frac{5}{3}} = \boxed{-\frac{2}{5}}$

Hint: Identify a and r.

(b) $\displaystyle\sum_{n=1}^{\infty} 4\left(\frac{1}{3}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n+1}}{2^n}$