Sequences and Limits

Definition: An **infinite sequence** is simply an infinitely long list of numbers. Each number in the sequences is called a **term** of the sequence.

Here is an example! We name this sequencee $\{a_n\}$:

$$\{a_n\} = \{2, 3, 5, 7, 11, 13, \ldots\}$$

$a_1 = 2,$ $a_4 = 7,$
$a_2 = 3,$ $a_5 = 11,$
$a_3 = 5,$ $\ldots$

The third term of this sequence is the number 5.

1. Find a formula for the n-th term of the following infinite sequences:

Go ahead and assume we start with $n = 1$. It's not always possible to find a formula, but for these examples you can.

(a) $\{a_n\} = \left\{e \cdot \pi, \frac{e\cdot\pi^2}{2}, \frac{e\cdot\pi^3}{3}, \frac{e\cdot\pi^4}{4}, \frac{e\cdot\pi^5}{5}, \ldots\right\}$ $\quad a_n = \frac{e\pi^n}{n}$

(b) $\{b_n\} = \{1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}, \ldots\}$ $\quad b_n = \frac{n}{2n-1}$

(c) $\{c_n\} = \{1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \ldots\}$ $\quad c_n = \frac{1}{n!}$

(d) $\{d_n\} = \{-1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots\}$ $\quad d_n = (-1)^n$

(e) $\{j_n\} = \{-1, \frac{1}{2}, \frac{-1}{6}, \frac{1}{24}, \frac{-1}{120}, \frac{1}{720}, \ldots\}$ $\quad j_n = (-1)^n \frac{1}{n!}$

(f) $\{k_n\} = \{3, 7, 3, 7, 3, 7, \ldots\}$ $\quad k_n = 5 + 2(-1)^n$

2. (a) For the sequence $\{a_n\}$ in Problem #1a, what is $\lim_{n\to\infty} a_n$? Does not exist (∞)

(b) For the sequence $\{b_n\}$ in Problem #1b, what is $\lim_{n\to\infty} b_n$? $\frac{1}{2}$

(c) For the sequence $\{c_n\}$ in Problem #1c, what is $\lim_{n\to\infty} c_n$? 0

(d) For the sequence $\{d_n\}$ in Problem #1d, what is $\lim_{n\to\infty} d_n$? does not exist

(e) For the sequence $\{j_n\}$ in Problem #1e, what is $\lim_{n\to\infty} j_n$? 0

(f) For the sequence $\{k_n\}$ in Problem #1f, what is $\lim_{n\to\infty} k_n$? does not exist

3. It's time to play the game "You make the definition"!!! Circle one choice for each.

Make sure you explain!

(a) The sequence $\{a_n\}$ converges / diverges.

(b) The sequence $\{b_n\}$ converges / diverges.

(c) The sequence $\{c_n\}$ converges / diverges.

(d) The sequence $\{d_n\}$ converges / diverges.

(e) The sequence $\{j_n\}$ converges / diverges.

(f) The sequence $\{k_n\}$ converges / diverges.

Sequence $\{a_n\}$ converges to L if terms of a_n get as close as you like to L when n is large enough.

4. For each of the following formulas for a_n, does the sequence $\{a_n\}$ converge or diverge? If it converges, what does it converge to?

(a) $a_n = \frac{27}{n^2-2}$

(b) $a_n = \sin(\pi n)$ — Careful! ☺

(c) $a_n = \frac{n+5}{3n-2}$

(d) $a_n = \cos(\pi n)$

5. **Phil:** Hey there, Renita! Have I shown you my favorite sequence? Here it is:

$$2.9, 2.95, 2.959, 2.9595, 2.95959, 2.959595, \ldots$$

Renita: Oh, I get it! I love patterns.

Phil: The problem is I am having trouble finding a formula for it, so I cannot figure out how to argue that this sequence converges, even though it clearly does!

Renita: The terms of your sequence are getting larger and larger, right?

Phil: You betcha!

Renita: And none of the terms of your sequence are larger than 3, right?

Phil: Yeah...

Renita: That's enough information to guarantee your sequence converges!

(a) Why does Renita say "that's enough information to guarantee your sequence converges"?

(b) Try to modify Renita's argument for the sequence

$$2.11, 2.012, 2.0013, 2.00014, 2.000015, 2.0000016, 2.00000017, \ldots$$

6. (a) Define a sequence *recursively* as follows: — A recursive sequence is defined in terms of itself.

$$a_1 = 1, \qquad a_2 = 1, \qquad a_n = a_{n-1} + a_{n-2} \text{ for } n > 2.$$

Write the first 12 terms of this sequence. — Have you seen this sequence before?

(b) Write the first ten terms of the sequence defined by $b_n = \dfrac{a_{n+1}}{a_n}$.

(c) Using the definition in part (b), show that $b_n = 1 + \dfrac{1}{b_{n-1}}$.

(d) Let $\lim_{n\to\infty} b_n = \rho$. Show that $\rho = 1 + \dfrac{1}{\rho}$, and solve this equation for ρ. — ρ is the "golden ratio"!

7. *Spicy!* Find the exact limit of the following sequence: — Find a formula for the sequence that is recursive.

$$\left\{\sqrt{2}, \sqrt{1+\sqrt{2}}, \sqrt{1+\sqrt{1+\sqrt{2}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}}, \ldots\right\}$$