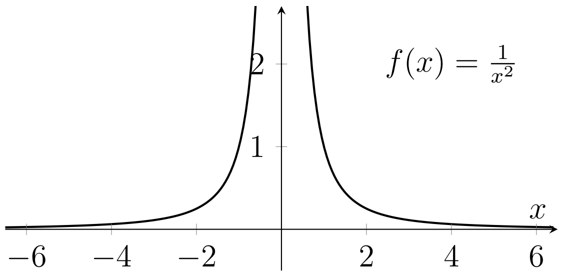
Improper Integrals

1. Here is a graph of the function $f(x) = \frac{1}{x^2}$:



(a) *Warm-up:* Compute each of the following. Write your answers as decimals.

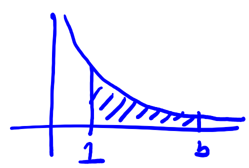
$$\int_1^{10} \frac{1}{x^2}\,dx = 0.9, \quad \int_1^{100} \frac{1}{x^2}\,dx = 0.99, \quad \int_1^{1000} \frac{1}{x^2}\,dx = 0.999$$

Now rephrase each of these integrals as a statement about area.

(b) Now use the FTC to find a formula for $\displaystyle\int_1^b \frac{1}{x^2}\,dx$.
Then rephrase this integral as a statement about area.

☞ You can pretend that b is some number larger than 1.

$$\int_1^b \frac{1}{x^2}\,dx = 1 - \frac{1}{b}$$



(c) Did you find $\displaystyle\int_1^b \frac{1}{x^2}\,dx = 1 - \frac{1}{b}$? (If not, check your work!) What happens when b is *really* big?

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = 1$$

(d) What is the area of the region under the graph of $f(x) = \frac{1}{x^2}$ between $x = 1$ and some *really* big b?

(e) *Discuss with your neighbors*: How would YOU define the integral

☞ Be careful! You cannot evaluate a function at ∞.

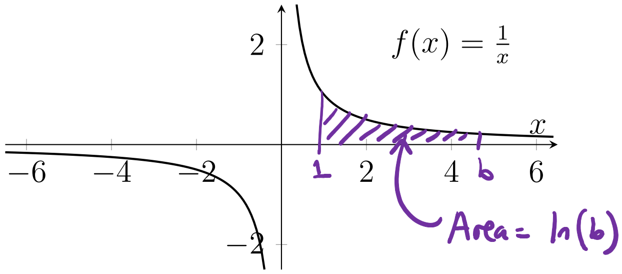
Improper Integral

$$\int_1^\infty \frac{1}{x^2}\,dx? = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$$

(f) *Discuss with your neighbors*: Now how would YOU define the integral

$$\int_{-\infty}^{-1} \frac{1}{x^2}\,dx? = \lim_{b\to-\infty} \int_b^{-1} \frac{1}{x^2}\,dx$$

2. Here is a graph of the function $f(x) = \frac{1}{x}$:

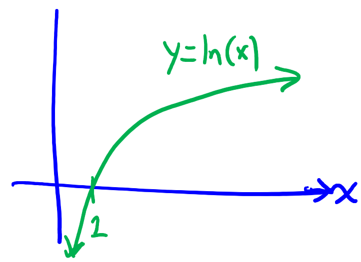


(a) Find a formula for $\displaystyle\int_1^b \frac{1}{x}\, dx$.

Again, pretend that b is some number larger than 1.

$$\int_1^b \frac{1}{x}\,dx = \ln x\Big|_{x=1}^{x=b} = \ln(b) - \cancel{\ln(1)}^{\,0} = \ln(b)$$

(b) Did you get $\displaystyle\int_1^b \frac{1}{x}\, dx = \ln(b)$? (If not, go back.) What happens when b is *enormous*?



(c) How would you evaluate the integral

$$\int_1^\infty \frac{1}{x}\, dx? = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(b)$$

DOES NOT EXIST

(∞)

Definition Alert: Suppose $f(x)$ is a function that is continuous for all $x \geq 1$ (or some other specific number). Then, we can define what it means to integrate where ∞ is one of the bounds (weird!). That is,

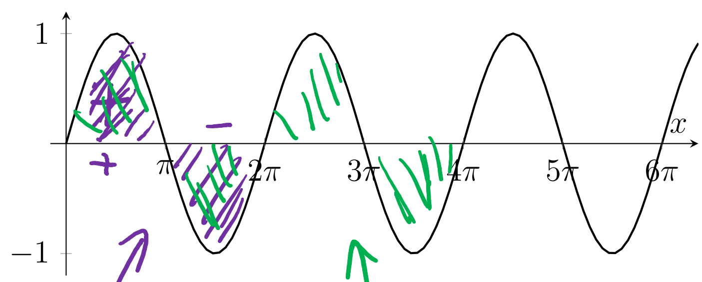
$$\int_1^\infty f(x)\, dx = \lim_{b\to\infty} \int_1^\infty f(x)\, dx.$$

This is called an **improper integral** because the function $f(x)$ isn't defined at $x = \infty$.

☞ If the limit exists, then we say that the integral **converges**.

☞ If the limit does *not* exist, then we say that the integral **diverges**.

3. Here is a graph of the function $f(x) = \sin(x)$:



(a) Without calculating any antiderivatives or doing any algebra, find each of the following:

$$\int_0^{2\pi} \sin(x)\,dx = 0 \qquad \int_0^{4\pi} \sin(x)\,dx = 0 \qquad \int_0^{6\pi} \sin(x)\,dx = 0$$

(b) *Discuss with your neighbors*: What do you think $\int_0^{\infty} \sin(x)\,dx$ should equal?

0?

(c) Use the FTC to evaluate $\int_0^{\pi} \sin(x)\,dx.$ $= -\cos(x)\Big|_0^{\pi} = -\cos(\pi) + \cos(0)$

$= 1 + 1 = 2$

☜ This is the area of one “bump” of the graph of $\sin(x)$.

(d) Without calculating any more antiderivatives or doing any more algebra, find each of the following:

$$\int_0^{3\pi} \sin(x)\,dx = 2, \quad \int_0^{5\pi} \sin(x)\,dx = 2$$

(e) *Discuss with your neighbors*: What do you think $\int_0^{\infty} \sin(x)\,dx$ should equal?

$$\int_0^{\infty} \sin(x)\,dx = \lim_{b\to\infty} \int_0^b \sin(x)\,dx = \lim_{b\to\infty} -\cos(x)\Big|_0^b = \lim_{b\to\infty}\left(-\cos(b)+1\right) \longleftarrow$$

oscillates, does not converge to any value

4. (a) Find a formula for $\int_a^1 \frac{1}{x^2}\,dx$. What happens when a gets *really* close to (but still just above) 0?

$$\int_a^1 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{x=a}^{x=1} = \frac{-1}{1} + \frac{1}{a} = \frac{1}{a} - 1$$

goes to ∞ as a goes to zero

(b) What do you think $\int_a^1 \frac{1}{x^2}\,dx$ should evaluate to?

$\int_a^1 \frac{1}{x^2}\,dx$ DIVERGES

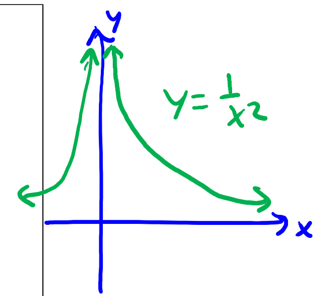
Definition Alert:

(i) If $f(x)$ is continuous for all x except $x = b$, then we define

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx.$$

(ii) If $f(x)$ is continuous for all x except $x = a$, then we define

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx.$$



$y = \frac{1}{x^2}$ is not continuous at $x = 0$

5. Does $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ converge or diverge? Note that $\frac{1}{x^2}$ is not continuous at $x=0$.

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}dx = \lim_{a\to 0^+}\left[\frac{-1}{x}\right]_{x=a}^{x=1} = \lim_{a\to 0^+}\left[-1+\frac{1}{a}\right]$$

and this limit does not exist, so the integral diverges

6. Does $\displaystyle\int_0^1 \frac{1}{x}\,dx$ converge or diverge?

$$\int_0^1 \frac{1}{x}dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\Big[\ln(x)\Big]_a^1 = \lim_{a\to 0^+}\big(\ln(1)-\ln(a)\big)$$

and this limit does not exist, so the integral diverges.

7. Does $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}\,dx$ converge or diverge?

$$\int_0^1 x^{-1/2}dx = \lim_{a\to 0^+}\int_a^1 x^{-1/2}dx = \lim_{a\to 0^+}\Big[2x^{\frac{1}{2}}\Big]_a^1 = \lim_{a\to 0^+}\left(2(1)^{1/2}-2(a)^{1/2}\right) = 2,$$

so the integral converges

8. Can we evaluate $\displaystyle\int_{-1}^1 \frac{1}{x}\,dx$? Here are two different arguments. Discuss with your neighbors the good and bad parts of each argument. Is either argument correct?

Think about this! We'll talk about it next time.

Argument 1: The function $\frac{1}{x}$, between $x = -1$ and $x = 1$, has one portion of area that counts "negative" and another portion of area that counts "positive." These areas are equal, and thus cancel, so the integral equals zero.

Argument 2: Integrals can be "split up", so

$$\int_{-1}^1 \frac{1}{x}\,dx = \int_{-1}^0 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx.$$

But, we just saw that $\displaystyle\int_0^1 \frac{1}{x}\,dx$ diverges on its own, so there is no hope when something else is added to it. So, the integral diverges.

9. Find the volume when the region under $f(x) = \frac{1}{x}$ between $x = 1$ and ∞ is rotated around the x-axis.