FUNDAMENTAL THEOREM EXAMPLE

If $f(x) = \int_{\cos x}^{x^2+5} \frac{\tan(t) - \ln(t)}{t^2 - t^7}\, dt$, then find $f'(x)$.

Method 1:

Let $g(t) = \frac{\tan(t) - \ln(t)}{t^2 - t^7}$, and let $G(t)$ be an antiderivative of $g(t)$.

Then: $f(x) = \int_{\cos x}^{x^2+5} g(t) = G(x^2+5) - G(\cos x)$ $\leftarrow$ by FTC 1

Differentiate: $f'(x) = 2x \cdot g(x^2+5) - (-\sin x)\, g(\cos x)$

$$f'(x) = 2x \frac{\tan(x^2+5) - \ln(x^2+5)}{(x^2+5)^2 - (x^2+5)^7} + \sin x \frac{\tan(\cos x) - \ln(\cos x)}{\cos^2 x - \cos^7 x}$$

Alternately: $f(x) = \int_{\cos x}^{x^2+5} \frac{\tan(t) - \ln(t)}{t^2 - t^7}\, dt$

$$f(x) = \int_0^{x^2+5} \frac{\tan(t) - \ln(t)}{t^2 - t^7}\, dt - \int_0^{\cos x} \frac{\tan(t) - \ln(t)}{t^2 - t^7}\, dt$$

By FTC 2: $f'(x) = 2x \frac{\tan(x^2+5) - \ln(x^2+5)}{(x^2+5)^2 - (x^2+5)^7} - (-\sin x) \frac{\tan(\cos x) - \ln(\cos x)}{\cos^2 x - \cos^7 x}$

Integration by Parts

1. (a) *Review*: Find the derivative of $f(x) = xe^x$.

$f'(x) = 1\,e^x + xe^x = e^x + xe^x$

PRODUCT RULE

(b) What is $\int (e^x + xe^x)\ dx$?

answer: $\int (e^x + xe^x)\ dx =$ $xe^x + C$

reason: since derivatives and integrals are opposite processes

(c) What is $\int e^x\ dx + \int xe^x\ dx$?

answer: $\boxed{\int e^x\ dx + \int xe^x\ dx = xe^x + C}$

reason: sum of integrals is the integral of the sum

(d) What is $\int xe^x\ dx$? $\int xe^x\,dx = xe^x + C - \int e^x dx = xe^x - e^x + C$

2. (a) *Review*: What is the general formula for $\dfrac{d}{dx}[u(x)v(x)]$?

PRODUCT RULE! $= u'(x)v(x) + u(x)v'(x)$

(b) What is $\int [u'(x)v(x) + u(x)v'(x)]\ dx$?

answer: $\int [u'(x)v(x) + u(x)v'(x)]\ dx =$ $u(x)v(x) + C$

reason: integral and derivative are opposites.

(c) What is $\int u'(x)v(x)\ dx + \int u(x)v'(x)\ dx$?

answer: $\int u'(x)v(x)\ dx + \int u(x)v'(x)\ dx =$ $u(x)v(x) + C$

reason: sum of integrals is integral of the sum

(d) Now can you devise a strategy to find $\int u(x)v'(x)\ dx$?

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx + C$$

dv is like $v(x)\,dx$

du is like $u(x)\,dx$

INTEGRATION BY PARTS

This method of integration uses this basic fact you hopefully just came up with:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

So, if you can identify one part of your desired integral as u and the *other* part of the integral as dv, this method will be successful ONLY IF the integral $\int v \cdot du$ is not worse than the one you started with!

Keep in mind a couple important pieces of strategy:

- You have to "break up" the integral into two multiplied pieces - one called u and the other called dv.
- Whatever you choose as u will have to have its derivative taken.
- Whatever you choose as v will have to have its antiderivative taken.
- You will have to find the antiderivative of $v \cdot du$ in order to finish.

3. Find $\int x \sin x \, dx$.

$\frac{du}{dx} = 1$

Guess for u: $u = x$ $\Rightarrow$ **Calculate du:** $du = dx$

Guess for dv: $dv = \sin(x)\,dx$ $\Rightarrow$ **Calculate v:** $v = -\cos x$

Hint: Your u and v' MUST multiply to equal $x \sin x.\,dx$

Now calculate: $uv - \int v \, du = x(-\cos x) - \int -\cos x \, dx = -x\cos x + \int \cos x \, dx$

$= \boxed{-x\cos x + \sin x + C}$

4. Find $\int x \ln x \, dx$.

Hint: You probably don't know the antiderivative of $\ln x$.

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$dv = x\,dx \Rightarrow v = \frac{1}{2}x^2$

$\int x \ln x = \underbrace{(\ln x)\frac{1}{2}x^2}_{u \cdot v} - \underbrace{\int \frac{1}{2}x^2 \frac{1}{x}\,dx}_{\int v\,du}$

$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\,dx$

$= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$

5. Find $\int x^3 \ln x \, dx$.

To be continued...

6. Find $\int \ln x \, dx$.

Hint: remember that $\ln x = 1 \cdot \ln x$.

7. Find $\int x^2 \cos x \, dx$.

8. Find $\int e^x \cos x \, dx$.

This one is spicy! You will need to do integration by parts *twice*, and you should end up with the same integral that you started with. It might seem hopeless, but is it?

9. Find $\int x^3 e^{x^2} \, dx$.

Slow down! Is there something easier than integration by parts for this?

10. Find $\displaystyle\int_0^{\pi} x \sin x \; dx$.

11. Find $\displaystyle\int_4^9 \frac{\ln y}{\sqrt{y}} \; dy$

12. Find $\displaystyle\int x^n \ln x \; dx$. Here, assume n is a constant and $n \neq 1$. (Why doesn't $n = 1$ work?)

13. Find $\displaystyle\int \frac{(\ln x)^2}{x} dx$.