The Substitution Rule

1. (a) *Warm-up/Review*: Find the derivative of $f(x) = \sin(x^2)$ and explain the method you use.

By the chain rule: $f'(x) = 2x\cos(x^2)$

(b) Now evaluate the indefinite integral $\int 2x\cos(x^2)\,dx$.

This is part (a) in reverse: $\int 2x\cos(x^2)\,dx = \sin(x^2) + C$

(c) Evaluate $\int 5x\cos(x^2)\,dx$. *Hint:* This function is 5/2 times the one in the previous problem.

$$\int 5x\cos(x^2)\,dx = \frac{5}{2}\int 2x\cos(x^2)\,dx = \frac{5}{2}\sin(x^2) + C$$

(d) Evaluate $\int 3x^2\cos(x^3)\,dx$. *Hint:* Look back at the relationship between parts (a) and (b).

Since $\frac{d}{dx}\left(\sin(x^3)\right) = 3x^2\cos(x^3)$, we find:

$$\int 3x^2\cos(x^3)\,dx = \sin(x^3) + C$$

The Substitution Rule: With the substitution $u = g(x)$:

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(u) + C = f(g(x)) + C$$

2. Evaluate the following indefinite integrals. For each substitution, clearly state your choice for u.

(a) $\int \underbrace{2x\,dx}_{du}\, e^{\overset{u}{x^2-5}} = \int e^u\,du = e^u + C = e^{x^2-5} + C$

$u = x^2 - 5$

$du = 2x\,dx$

(b) $-\frac{1}{2}\int \underbrace{-2x\,dx}_{du}\, \sin(\overset{u}{1-x^2}) = -\frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\left(-\cos(u)\right) + C = \frac{1}{2}\cos(1-x^2) + C$

$u = 1 - x^2$

$du = -2x\,dx$

CHECK YOUR ANSWERS BY DIFFERENTIATING!

3. Chassidy and Dave are having a conversation about the integral

$$\int \sin(x)e^{\cos(x)}\, dx$$

Dave: OK. Let's try substitution for this one!

Chassidy: Why do you think substitution will work? We are supposed to find something to make u and then search for the derivative of u, also. Oh, I guess we can do that!

Dave: Yeeeeeeah! We can make u be $\sin(x)$ and I see $du = \cos(x)$ is also in there. Easy!

Chassidy (shaking her head): Dave, Dave, Dave...

(a) Why is Chassidy shaking her head at Dave?

$\cos(x)$ is in an exponent, and we don't want to put du in an exponent.

(b) What would be a more useful choice for u? Why?

Try $u = \cos(x)$. Then $du = -\sin(x)\, dx$, which we see in the integral (and not in an exponent).

After explaining her idea to Dave, Chassidy and Dave are left with the problem

$$\int -e^{u}\, du$$

Chassidy: I can do that integral!! The final answer is $-e^{u} + C$. Yes!!

Dave (shaking his head): My turn! Chassidy, Chassidy, Chassidy...

(c) Why is Dave shaking his head at Chassidy?

The answer should be given in terms of the original variable x, not u.

The answer is: $-e^{\cos x} + C$

Substitution for definite integrals

Either:

(a) Compute the indefinite integral, expressing an antiderivative in terms of the original variable, then evaluate the result at the original bounds,

Or:

(b) Convert the original bounds of integration to new bounds in terms of the new variable, and do not convert the antiderivative back to the original variable.

4. Evaluate the following definite integrals. (Some of the functions inside of the integrals should look familiar!)

Recall the indefinite integrals on Page 1.

(a) $\displaystyle\int_{x=0}^{x=\sqrt{\pi}} 2x\cos(x^2)\,dx = \sin(x^2)\Big|_{x=0}^{x=\sqrt{\pi}} = \sin(\pi) - \sin(0) = 0$

$u = x^2$

OR: $\displaystyle = \int_{u=0}^{u=\pi} \cos(u)\,du = \sin(u)\Big|_{u=0}^{u=\pi} = \sin(\pi) - \sin(0) = 0$

(b) $\displaystyle\int_{x=0}^{x=3} 2xe^{x^2-5}\,dx = e^{x^2-5}\Big|_{x=0}^{x=3} = e^4 - e^{-5}$

$u = x^2 - 5$

OR: $\displaystyle = \int_{u=-5}^{u=4} e^u\,du = e^u\Big|_{u=-5}^{u=4} = e^4 - e^{-5}$

(c) $\displaystyle\frac{1}{9}\int_{x=0}^{x=1} \frac{9x^2}{2+3x^3}\,dx = \frac{1}{9}\int_{u=2}^{u=5} \frac{1}{u}\,du = \frac{1}{9}\ln|u|\Big|_{u=2}^{u=5} = \frac{1}{9}\left(\ln 5 - \ln 2\right) = \frac{1}{9}\ln\left(\frac{5}{2}\right)$

$u = 2 + 3x^3$

$du = 9x^2\,dx$

(d) $\displaystyle -\frac{1}{2}\int_{x=0}^{x=3} -2x\sqrt{9-x^2}\,dx = -\frac{1}{2}\int_{u=9}^{u=0}\sqrt{u}\,du = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_{u=9}^{u=0} = -\frac{1}{3}\left(0^{3/2} - 9^{3/2}\right) = -\frac{1}{3}(-27) = 9$

$u = 9 - x^2$

$du = -2x\,dx$

5. If $\displaystyle\int_1^4 f(x)\,dx = 5$, evaluate $\displaystyle\int_0^1 f(3x+1)\,dx$.

$\displaystyle\frac{1}{3}\int_{x=0}^{x=1} 3f(3x+1)\,dx = \frac{1}{3}\int_1^4 f(u)\,du = 5$

$u = 3x + 1$

$du = 3\,dx$