CYCLES OF DIGITS

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The decimal representations of the multiples of $\frac{1}{7}$ display a striking pattern:

$$\frac{1}{7} = 0.142857142857\ldots \qquad \frac{4}{7} = 0.571428571428\ldots$$
$$\frac{2}{7} = 0.285714285714\ldots \qquad \frac{5}{7} = 0.714285714285\ldots$$
$$\frac{3}{7} = 0.428571428571\ldots \qquad \frac{6}{7} = 0.857142857142\ldots$$

These multiples of $\frac{1}{7}$ contain *cyclic permutations of digits*—that is, the same repeating sequence of six digits, each multiple starting with a different digit. More specifically, for $m = 1, 2, \ldots, 6$, the first digit after the decimal place in the representation of $\frac{m}{7}$ is the m^{th} smallest digit in the repeating sequence 142857.

This seems remarkable. At the very least, knowledge of this pattern allows one to quickly determine the decimal representation of any multiple of $\frac{1}{7}$. Yet the pattern prompts us to ask many questions. Why does this pattern exist? Do fractions with other denominators exhibit similar patterns? What happens in bases other than ten? A bit of abstract algebra can help answer these questions. Moreover, repeating decimals provide a case study that can help illuminate important concepts in abstract algebra.

Long Division

Suppose we want to calculate by hand the decimal representation of $\frac{1}{7}$, doing long division as in Figure 1 on the next page. In the quotient, we obtain the repeating sequence 142857, which is called the *reptend* [5]. In order to understand where the reptend comes from, focus instead on the sequence of highlighted numbers in the long division: $1, 3, 2, 6, 4, 5, 1$. This sequence consists of the first digit of the dividend, followed by the *remainders* at each step of the long division. When we reach a remainder of 1, the decimal begins to repeat. For $\frac{1}{7}$, we reach a 1 on the sixth remainder, which is why the reptend is six digits long. We call this length the *period* of $\frac{1}{7}$.

In fact, for $\frac{1}{7}$ this period is as long as it could possibly be. When dividing 1 by 7, there are only six possible remainders: the integers from 1 to 6. (We will never get a remainder of zero, because $\frac{1}{7}$ is not a terminating decimal.) As soon as we reach a remainder that we have already seen, the decimal begins to repeat. Since there are only six possible remainders, we can't go any more than six steps without seeing one of them twice.

We say that a fraction $\frac{1}{p}$ has *maximal period* if its period is $p - 1$; that is, if its period is as long as it could possibly be. Thus, $\frac{1}{7}$ has maximal period.

Desiring to better understand this sequence of remainders, we ask: Where do these numbers come from? One answer is that they are simply the remainders we get when we divide 1 by 7. A

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$$
\begin{array}{l}
\quad\; 0.142857\ldots \\
7\,\overline{)\,\mathbf{1}.0000000\ldots} \\
\quad\; \underline{-7} \\
\qquad \mathbf{3}0 \\
\quad\;\; \underline{-28} \\
\qquad\;\; \mathbf{2}0 \\
\qquad \underline{-14} \\
\qquad\quad \mathbf{6}0 \\
\qquad\;\; \underline{-56} \\
\qquad\qquad \mathbf{4}0 \\
\qquad\quad \underline{-35} \\
\qquad\qquad\; \mathbf{5}0 \\
\qquad\qquad \underline{-49} \\
\qquad\qquad\quad \mathbf{1}0 \\
\qquad\qquad\qquad \ddots
\end{array}
$$

FIGURE 1. Computing the decimal representation of $\frac{1}{7}$ via long division.

more illuminating answer, as we will see, is that these are the powers of 10, reduced modulo 7. To better understand this, we must discuss the *units group* of 7.

THE UNITS GROUP

The abstract algebra that will help us understand decimal representations involves something called the *units group* of a positive integer n. For any positive integer n, let U_n denote the set of all positive integers less than n and relatively prime to n. For example, if $n = 9$, this set is $U_9 = \{1, 2, 4, 5, 7, 8\}$. This set forms a *group* under multiplication modulo n. This means that U_n satisfies the group axioms:

(1) **Closure**: The set U_n is *closed* under multiplication modulo n. That is, the product of any two elements of U_n, reduced modulo n, is an element of U_n. In U_9, for example, $4 \cdot 8 = 32$; reduced modulo 9 this is equivalent to 5, which is in U_9.
(2) **Associativity**: Multiplication modulo n is *associative*, meaning that for any $a, b, c \in U_n$, we have $a(bc) = (ab)c$, with each multiplication computed modulo n. This follows from the associativity of multiplication of integers.
(3) **Identity**: There is an *identity* element in U_n, namely the number 1, such that $1a = a1 = a$ for all a in U_n.
(4) **Invertibility**: For any element a in U_n, there exists an element b in U_n, such that $ab \equiv 1 \pmod{n}$. Such an element b is called the *inverse* of a. In U_9, for example, if $a = 4$, then we find $b = 7$, and we see that $4 \cdot 7 = 28 \equiv 1 \pmod{9}$.

Indeed, the invertibility property gives the group its name. The reader who has studied ring theory may recall that the set $\{0, 1, 2, \ldots, n-1\}$, with addition and multiplication modulo n, forms a *ring* denoted $\mathbb{Z}/n\mathbb{Z}$. The elements that have multiplicative inverses in a ring are called *units* of the ring. Thus, U_n is the group of units in the ring $\mathbb{Z}/n\mathbb{Z}$. Furthermore, the units in U_n are precisely the positive integers that are less than n and relatively prime to n.

It is also important to keep straight the concept of *order*, both of a group and of an element in a group. The *order of a group* is the number of elements in the group and is denoted by vertical bars around the name of the group, such as $|U_n|$. The *order of an element* a in U_n is the smallest positive integer r such that $a^r \equiv 1 \pmod{n}$. The order of an element a is denoted $|a|$. For example, $|U_9| = 6$

because U_9 has 6 elements. In U_9, $|4| = 3$ because $4^3 = 64 \equiv 1 \pmod 9$, and no smaller positive power of 4 is congruent to 1 (modulo 9).

Understanding Repeating Decimals

Let's turn back to the sequence of remainders in the long division of 1 by 7, highlighted in Figure 1. The sequence of remainders is really the sequence of powers of 10 in U_7. The powers of 10, reduced modulo 7, are as follows:

$$10^1 \equiv 3, \quad 10^2 \equiv 2, \quad 10^3 \equiv 6, \quad 10^4 \equiv 4, \quad 10^5 \equiv 5, \quad 10^6 \equiv 1, \ldots$$

We see that the order of 10 in U_7 is 6, because 10^6 is the first power of 10 to be congruent to 1 (modulo 7). This is why it takes *six* steps of the long division before we encounter a remainder of 1 (indeed, before we encounter *any* repeated remainder). We say that 10 (or equivalently, 3, since we're working modulo 7) *generates* U_7 because every element of U_7 is congruent to some power of 10 (modulo 7).

Similar reasoning holds for $\frac{1}{n}$ whenever n is relatively prime to 10; that is, whenever n does not have any factors of 2 or 5. More formally, we have the following theorem:

Theorem. *If* 10 *is relatively prime to* n*, then the period of* $\frac{1}{n}$ *is equal to the order of* $10 \pmod n$ *in* U_n*.*

Proof. If the period of $\frac{1}{n}$ is ℓ, then this means that ℓ is the smallest positive integer such that

$$\frac{10^\ell - 1}{n}$$

is an integer. (Consider the trick for converting a repeating decimal to a fraction—multiplying by 10^ℓ and subtracting off the decimal part.) This means that ℓ is the smallest positive integer such that n divides $10^\ell - 1$, or equivalently, $10^\ell \equiv 1 \pmod n$. But this last statement is precisely the *definition* of the order of 10 in U_n. □

Exercise 1. Find the period of $\frac{1}{13}$ by finding $|10|$, the order of 10, in U_{13}. Check your answer by computing the decimal representation of $\frac{1}{13}$ via long division. Observe that the remainders you encounter in the long division correspond to the powers of 10 in U_{13}.

Full Reptend Primes

Now that we know how to find the period of $\frac{1}{n}$ for n relatively prime to 10, we can understand the cyclic permutations exhibited by the *multiples* of some fractions like $\frac{1}{7}$. It turns out that cyclic permutations of digits occur in the multiples of many fractions $\frac{1}{p}$, where p is prime.

Suppose p is an odd prime and the order of 10 in U_p is $p-1$. This means that all of the numbers $1, 2, \ldots, p-1$ appear as powers of 10 in U_p. That is, for each $m \in \{1, 2, \ldots, p-1\}$, we can find some positive integer r such that $m \equiv 10^r \pmod p$. Since $m \equiv 10^r \pmod p$, the *fractional parts* (i.e. the digits to the right of the decimal point) of $\frac{m}{p}$ and $\frac{10^r}{p}$ are the same. This means that $\frac{m}{p}$ has the same sequence of repeating digits as $\frac{1}{p}$, but starting with a different digit.

Example. We can use $p = 7$ as an example. We saw earlier that $\frac{1}{7} = 0.\overline{142857}$ (the overbar indicates that the digits repeat). Let's replace the numerator with a different number, say 6. Now $6 \equiv 10^3 \pmod 7$, so the fractional parts of $\frac{6}{7}$ and $\frac{10^3}{7}$ are the same. Since we know the decimal representation of $\frac{1}{7}$, we simply shift the decimal point to find that $\frac{10^3}{7} = 142.\overline{857142}$. Thus, $\frac{6}{7} = 0.\overline{857142}$, which is the same sequence of repeating digits as in $\frac{1}{7}$, but starting with a different digit. We can illustrate this pattern in a circle diagram as in Figure 2, inspired by Brenton and Pineau [1, 6].

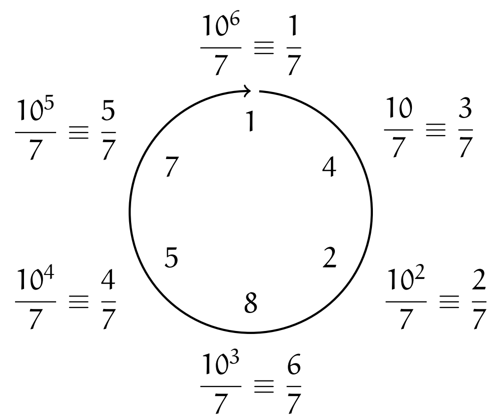

FIGURE 2. Circle diagram for $\frac{1}{7}$: The digits in the decimal representation of $\frac{1}{7}$ are inside the circle. The multiples of $\frac{1}{7}$ and their corresponding powers of 10 appear outside the circle, placed next to the digit that begins each repeating sequence.

When p is prime, *all* positive integers less than p are relatively prime to p, so $U_p = \{1, 2, \ldots, p-1\}$. A basic theorem in abstract algebra says that the order of an element in a group always divides the order of the group itself. This means that the order of 10 in U_p, and thus the period of $\frac{1}{p}$, always divides $p-1$.

For $p = 7$, the order of 10 in U_7 is 6, which is as large as it could possibly be. Of course, this corresponds to the previously-observed fact that the period of $\frac{1}{7}$ is as large as it could possibly be. Mathematicians have a special name for primes like 7: they are called *full reptend primes*.[1]

Definition. A prime p is a *full reptend prime* if the order of 10 in U_p is $p-1$.

In summary, a prime p is full reptend if and only if $|10| = p-1$ in U_p, which is the case if and only if the decimal representation of $\frac{1}{p}$ has period $p-1$. This also occurs exactly when 10 generates U_p. Furthermore, for any full reptend prime, the multiples $\frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}$ exhibit cyclic permutations of a sequence of $p-1$ digits.

Exercise 2. Find the order of 10 in U_{13} and also in U_{17}. (*Hint*: refer to Exercise 1.) Which of 13 and 17 is a full reptend prime?

We are beginning to see that the cyclic permutations exhibited by multiples of $\frac{1}{7}$, while remarkable, are not rare. In fact, the first few full reptend primes are:

$$7, 17, 19, 23, 29, 47, 59, 61, 97, 109, 113, \ldots$$

The multiples of the reciprocals of *each* of these primes contain cyclic permutations of digits. This prompts us to ask more questions, such as: How many full reptend primes are there?

Many mathematicians think that there are infinitely many full reptend primes, but no one has proved this yet. More specifically, Emil Artin conjectured that about 37% of all primes are full reptend primes. In fact, Artin was quite precise: he conjectured that the set of full reptend primes has asymptotic density in the set of all primes equal to *Artin's constant*, denoted C and defined by the infinite product:

$$C = \prod_{p \text{ prime}} \left(1 - \frac{1}{p(p-1)}\right) = 0.373955\ldots$$

[1]These primes also go by other names, including *full-period primes* and *long primes*.

Basically, this means that if we examine more and more primes (in ascending order, starting with the smallest prime), then we will find that the percentage of full reptend primes approaches C as the number of primes we examine goes to infinity. For more information, see Conway and Guy [2] or Shanks [7].

PRIMES LESS THAN FULL

What about primes that are *not* full reptend, such as 13? The order of 10 in U_{13} is 6, so we know that the period of $\frac{1}{13}$ is 6. Do the multiples of $\frac{1}{13}$ exhibit similar cyclic patterns as the full reptend primes? The answer is *yes*—and furthermore, these cyclic patterns provide an illustration of *cosets* in a group.

Consider the multiples of $\frac{1}{13}$:

$$\frac{1}{13} = 0.\overline{076923} \qquad \frac{5}{13} = 0.\overline{384615} \qquad \frac{9}{13} = 0.\overline{692307}$$
$$\frac{2}{13} = 0.\overline{153846} \qquad \frac{6}{13} = 0.\overline{461538} \qquad \frac{10}{13} = 0.\overline{769230}$$
$$\frac{3}{13} = 0.\overline{230769} \qquad \frac{7}{13} = 0.\overline{538461} \qquad \frac{11}{13} = 0.\overline{846153}$$
$$\frac{4}{13} = 0.\overline{307692} \qquad \frac{8}{13} = 0.\overline{615384} \qquad \frac{12}{13} = 0.\overline{923076}$$

Six of the above numbers contain the repeating sequence 076923, and the other six contain the repeating sequence 153846. Why is this?

The six numbers that contain the repeating sequence 076923 are the fractions whose numerators are congruent to powers of 10 (modulo 13). Since $10^6 \equiv 1 \pmod{13}$, we see that 10 has order 6 in U_{13}. The set of powers of 10 is the *subgroup generated by* 10 in U_{13}. This subgroup is denoted by $\langle 10 \rangle$. In other words, $\langle 10 \rangle = \{10, 9, 12, 3, 4, 1\}$ in U_{13}. Thus, the fractions $\frac{1}{13}$, $\frac{10}{13}$, $\frac{9}{13}$, $\frac{12}{13}$, $\frac{3}{13}$, and $\frac{4}{13}$ all contain the repeating sequence of digits 076923, illustrated in the circle diagram on the left in Figure 3.

The other six multiples of $\frac{1}{13}$, which contain the repeating sequence 153846, are related to a *coset* of $\langle 10 \rangle$ in U_{13}. Observe that 2 is not in $\langle 10 \rangle$. If we multiply each element of $\langle 10 \rangle$ by 2, we get the

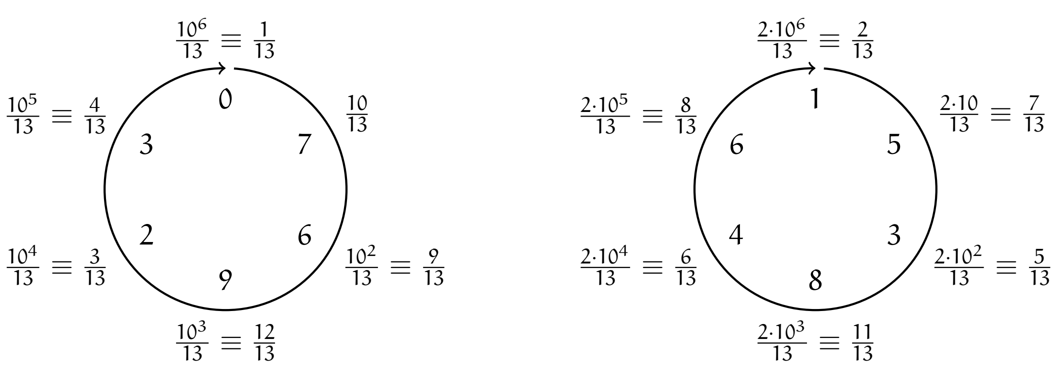

FIGURE 3. Circle diagrams for $\frac{1}{13}$: The multiples of $\frac{1}{13}$ form two sets, each set displaying cyclic permutations of a sequence of six digits. The numerators around the left circle comprise $\langle 10 \rangle$ in U_{13}, while the numerators around the right circle are those in $2\langle 10 \rangle$.

following subset of U_{13}, which we denote by $2\langle 10\rangle$:

$$2\langle 10\rangle = \{7, 5, 11, 6, 8, 2\}$$

Notice that this set contains all the elements of U_{13} *not* in $\langle 10\rangle$. This set is called a *coset* of $\langle 10\rangle$ in U_{13}. In fact, we obtain the same set if we multiply the elements of $\langle 10\rangle$ by any element of U_{13} that isn't already in $\langle 10\rangle$.

The elements of this coset correspond to six multiples of $\frac{1}{13}$ whose decimal representations share the repeating sequence of digits 153846. For example, $\frac{2}{13} = 0.\overline{153846}$. Multiplying by 10, we find $\frac{20}{13} = 1.\overline{538461}$, and we restrict our attention to the fractional part to obtain $\frac{7}{13} = 0.\overline{538461}$. Similarly, any multiple of $\frac{1}{13}$ with numerator in $2\langle 10\rangle$ has the repeating sequence 153846 in its decimal representation. These multiples are illustrated in the circle diagram on the right in Figure 3.

We now see that the multiples of $\frac{1}{13}$ form two sets, each exhibiting cyclic permutations of a sequence of six digits. This is because the subgroup $\langle 10\rangle$ has two cosets in U_{13}: one coset is $\langle 10\rangle$ itself, and the other coset consists of the rest of U_{13} not in $\langle 10\rangle$. The numerators of fractions around the left circle in Figure 3 are the elements of $\langle 10\rangle$, while the numerators of fractions around the right circle in Figure 3 are the elements of $2\langle 10\rangle$.

In general, cosets partition a group into subsets of equal size. If prime p is full reptend, then $\langle 10\rangle$ has no cosets besides itself in U_p. If prime p is not full reptend, then the cosets of $\langle 10\rangle$ in U_p partition the group U_p into $\frac{p-1}{|10|}$ subsets of size $|10|$. Likewise, the multiples $\frac{1}{p}, \ldots, \frac{p-1}{p}$ can be partitioned into $\frac{p-1}{|10|}$ subsets of size $|10|$. Within each subset are multiples of $\frac{1}{p}$ that display cyclic permutations of the same sequence of digits.

Example. In U_{37}, $|10| = 3$, so the the decimal representation of $\frac{1}{37}$ repeats after only three digits. Indeed, $\frac{1}{37} = 0.\overline{027}$. Since U_{37} has 36 elements, and each coset of $\langle 10\rangle$ contains 3 elements, there must be 12 cosets. Thus, the multiples of $\frac{1}{37}$ display cyclic permutations of 12 different 3-digit sequences, as illustrated in Figure 4.

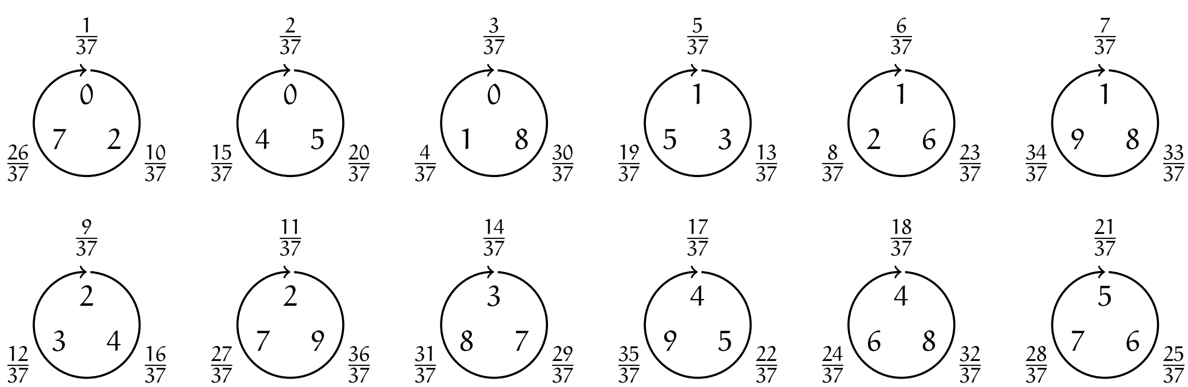

FIGURE 4. Circle diagrams for $\frac{1}{37}$: Each of the 12 cyclic patterns in the multiples of $\frac{1}{37}$ corresponds to a coset of $\langle 10\rangle$ in U_{37}.

Exercise 3. How many cosets does $\langle 10\rangle$ have in U_{41}? What cyclic permutations are exhibited by the multiples of $\frac{1}{41}$?

COMPOSITE DENOMINATORS

Suppose n is a non-prime positive integer. If n has no factor of 2 or 5, then $10 \in U_n$. In this case, U_n reveals information about cyclic permutations in *some* of the fractions $\frac{1}{n}, \dots, \frac{n-1}{n}$. However, U_n tells us nothing about those fractions with numerators not relatively prime to n. That is, if m is not relatively prime to n, then $m \notin U_n$, and the fraction $\frac{m}{n}$ is not in lowest terms. We must first reduce this fraction and examine a different units group to learn about its period.

Example. Let $n = 21$. Then $U_{21} = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$. The order of 10 is 6, and $\langle 10 \rangle = \{10, 16, 13, 4, 19, 1\}$. Thus, $\frac{1}{21}$ and five other multiples (those with numerators in $\langle 10 \rangle$) exhibit cyclic permutations of the same sequence of 6 digits.

Furthermore, $\langle 10 \rangle$ has one other coset in U_{21}, which means that six other multiples of $\frac{1}{21}$ display cyclic permutations of another sequence of 6 digits. The remaining multiples of $\frac{1}{21}$ include those that are actually multiplies of $\frac{1}{7}$, which we already considered, and those that are multiples of $\frac{1}{3}$. Figure 5 illustrates the circle diagrams for multiples of $\frac{1}{21}$.

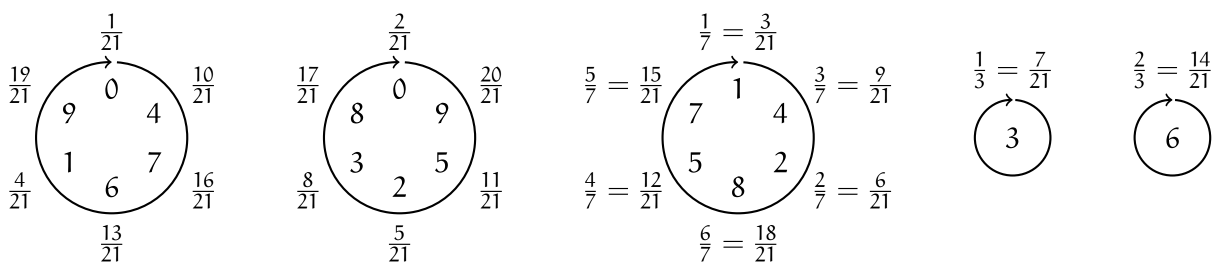

FIGURE 5. Circle diagrams for $\frac{1}{21}$: Since 21 is composite, we also must consider its factors. The two left circles correspond to cosets of $\langle 10 \rangle$ in U_{21}, the middle circle corresponds to U_7, and the two right circles correspond to single-element cosets in U_3.

Thus far, we have avoided considering any n with a factor of 2 or 5. Since 2 and 5 are the prime factors of 10, the base of the decimal system, we must treat with special care fractions whose denominator has a factor of 2 or 5. If n has *only* factors of 2 and 5, then the decimal representation of $\frac{1}{n}$ terminates. Otherwise, factors of 2 or 5 produce some non-repeating digits after the decimal point, before the repeating digits.

For any positive integer n, we can remove all factors of 2 and 5, writing $n = 2^a 5^b q$, where q and 10 are relatively prime. We can use our understanding of $\frac{1}{q}$ to help us understand $\frac{1}{n}$. Let $c = \max\{a, b\}$, and then write:

$$\frac{1}{n} = \frac{1}{2^a 5^b q} = \frac{1}{10^c} \cdot \frac{2^{c-a} 5^{c-b}}{q}.$$

Multiplying by $\frac{1}{10^c}$ just shifts the decimal point c places to the left. Thus, the behavior of $\frac{1}{n}$ can be found by studying the multiples of $\frac{1}{q}$ and, more specifically, U_q.

Example. Let $n = 140$, which we can factor as $n = 2^2 \cdot 5 \cdot 7$. Observing that

$$\frac{1}{140} = \frac{1}{10^2} \cdot \frac{5}{7},$$

we see that the decimal representation of $\frac{1}{140}$ has the same repeating sequence as $\frac{5}{7}$, with two zeros after the decimal point. That is, $\frac{1}{140} = 0.00\overline{714285}$.

Exercise 4. Referring to Figure 3 above, and without using a calculator, find the decimal representations of the following fractions: $\frac{1}{325}$, $\frac{1}{5200}$, and $\frac{9}{1625}$. (*Hint*: $325 = 25 \cdot 13$, $5200 = 16 \cdot 25 \cdot 13$, and $1625 = 125 \cdot 13$)

OTHER BASES

Cyclic permutations appear in other bases, too. Recall that the value of any digit in a number is determined by its *place*. In base ten, the value of any place is a power of ten. For a simple example,

$$265.41 = 2 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0 + 4 \cdot 10^{-1} + 1 \cdot 10^{-2}.$$

However, if "265.41" is interpreted as a base-b number, then its value is:

$$265.41 = 2 \cdot b^2 + 6 \cdot b^1 + 5 \cdot b^0 + 4 \cdot b^{-1} + 1 \cdot b^{-2}.$$

Note that each *digit* in a base b number must be less than b itself.

To avoid confusion, we use a subscript following a number to indicate the base in which the number is written. We may omit the subscripts for base-ten numbers when the base is clear. In this paper, fractions will always indicate a quotient of two base-ten numbers. For instance,

$$265.41_7 = 2 \cdot 7^2 + 6 \cdot 7^1 + 5 \cdot 7^0 + 4 \cdot 7^{-1} + 1 \cdot 7^{-2} = 145 + \frac{29}{49}.$$

Exercise 5. Why is 49_6 improperly expressed in base six? Check to make sure you understand each of the following base conversions:

$$253_7 = 136_{10}, \qquad 0.102_3 = \frac{11}{27}, \qquad \text{and} \qquad 14.3_8 = 12.375_{10}.$$

Just as in base ten, the units group U_n indicates what cyclic permutations appear when the multiples of $\frac{1}{n}$ are written in base b, whenever n and b are relatively prime. We simply look at the behavior of b in the group U_n. In particular, the multiples of $\frac{1}{p}$ display maximal-period cyclic permutations in base b, if and only if b is a generator of U_n.

Example. For example, consider the base-8 representation of $\frac{1}{5}$. Since $8 \equiv 3 \pmod 5$, we examine the behavior of 3 in U_5:

$$3^1 \equiv 3, \quad 3^2 \equiv 4, \quad 3^3 \equiv 2, \quad \text{and} \quad 3^4 \equiv 1.$$

In U_5, $|8| = 4$, which is as large as it could be. This means that the multiples of $\frac{1}{5}$ display maximal-period cyclic permutations when expressed in base 8. Explicitly, these multiples are:

$$\frac{1}{5} = 0.\overline{1463}_8, \quad \frac{2}{5} = 0.\overline{3146}_8, \quad \frac{3}{5} = 0.\overline{4631}_8, \quad \text{and} \quad \frac{4}{5} = 0.\overline{6314}_8.$$

We say that 5 is a *full reptend prime in base* 8 since 8 has maximal order in U_5. The following theorem summarizes the connection between full reptend primes and cyclic permutations in any base.

Theorem. *The period of the base-b representation of $\frac{1}{p}$ is equal to $|b|$ in U_p. Each coset of $\langle b \rangle$ in U_p corresponds to a set of multiples of $\frac{1}{p}$ that display cyclic permutations of digits in their base-b representations. In particular, the base-b representations of $\frac{1}{p}, \ldots, \frac{p-1}{p}$ contain maximal-period cyclic permutations of digits if and only if b is a generator of U_p, which is the case exactly when p is a full reptend prime in base b.*

For any prime p, there are bases in which p is a full reptend prime. Furthermore, mathematicians think that for *any* base b, there exist infinitely many full reptend primes in base b, but this hasn't been proven for even a single base! For more information about this, see Chapter 6 of Conway and Guy [2].

Exercise 6. In which bases is 13 a full reptend prime? For an extra challenge, compute the decimal representation of $\frac{1}{13}$ in one such base.

Conclusion

Cyclic permutations of digits are not unusual, but occur in the decimal representations of many fractions. Indeed, it seems that there are infinitely many full reptend primes, the reciprocals of which display maximal-period cyclic permutations. Perhaps the main reason why we do not often observe these cyclic permutations is because we hardly ever write out enough digits. The cyclic permutations of the multiples of $\frac{1}{17}$ may be virtually unknown because we almost never compute decimals to 16 digits. How much more rarely observed are the patterns that appear in the reciprocals of the larger full reptend primes!

A little abstract algebra goes a long way in explaining cyclic permutations of digits. This illumination itself may be of a reciprocal nature, as understanding cyclic permutations may help the student grasp concepts in abstract algebra. It is the author's hope that the reader who has made it this far has an increased appreciation, not only for patterns of digits, but also for the units group U_n.

The story does not end here, for many problems remain to be solved. Are there really infinitely many full reptend primes? Are there infinitely many in all bases? What is the most efficient way of finding full reptend primes in a given base? While easy to understand, the solutions to these problems likely involve very sophisticated mathematics. The interested reader may wish to try the following exercies or pursue further study of number theory and abstract algebra (for possible resources, see the texts by Vanden Eynden [3] and Gallian [4]).

Exercise 7. A number written with only the digit one, such as 1111, is called a *repunit* (short for "repeated unit"). Prove that the period of $\frac{1}{p}$ is equal to the number of digits in the first repunit in the sequence $11, 111, 1111, \ldots$ that is divisible by p.

Exercise 8. Notice that diametrically opposite numbers inside of the circles in Figure 2 and Figure 3 always sum to 9. This is known as *Midy's theorem*: if $\frac{1}{p}$ has even period, then the digits of the reptend can be split into two halves whose sum is a string of 9s. For example, $\frac{1}{7} = 0.\overline{142857}$ and $142 + 857 = 999$.

(a) Can you prove that Midy's theorem holds for any prime p such that $\frac{1}{p}$ has even period?
(b) Does a version of Midy's theorem hold in base b? Why or why not?
(c) How Midy's theorem equivalent to the observation that diametrically opposite fractions outside the circles in diagrams such as Figure 2 always sum to 1?

References

[1] Lawrence Brenton. "Remainder Wheels and Group Theory." *College Math Journal*, 39 (2008) pp. 129-135.
[2] John H. Conway and Richard K. Guy. *The Book of Numbers*. Springer-Verlag, 1996, pp. 157-171.
[3] Charles Vanden Eynden. *Elementary Number Theory*, 2nd ed. McGraw-Hill, 2001.
[4] Joseph A. Gallian. *Contemporary Abstract Algebra*, 8th ed. Cengage, 2012.
[5] Dan Kalman. "Fractions with Cyclic Digit Patterns." *College Math Journal*, 27 (1996) pp. 109-115.
[6] Richard Pineau. "What Goes Around Comes Around." *Math Horizons*, Nov. 2012, pp. 22-24.
[7] Daniel Shanks. *Solved and Unsolved Problems in Number Theory*, 4th ed. American Mathematical Society, 2001, pp. 80-83.